THE MAY MEETING IN NEW YORK

The two hundred forty-second regular meeting of the Society was held at Columbia University, on Saturday, May 2, 1925, extending through the usual morning and afternoon sessions. The attendance included the following sixty-three members.

Alexander, R. L. Anderson, C. R. Ballantine, J. P. Ballantine, Boyajian, Brant, R. W. Burgess, Carson, Alonzo Church, Dostal, Dresden, Eversull, Feldstein, Fenn, Fine, Fiske, Fite, D. A. Flanders, Fort, R. M. Foster, Philip Franklin, Frink, Gafafer, Garretson, Gehman, Gill, Glenn, W. C. Graustein, Gray, Gronwall, C. C. Grove, Hazlett, Hille, Himwich, Hoyt, Joffe, Kerékjártó, Kline, Lefschetz, Harry Levy, MacColl, S. P. Mead, Meder, Metz, Molina, Paxton, Pell, R. G. Putnam, Ritt, R. B. Robbins, Saurel, Schelkunoff, Seely, J. H. Taylor, J. M. Thomas, Tracey, Veblen, Wedderburn, Whited, Widder, Widmark, Wiener, Zobel.

There was no meeting of the Council or of the Trustees. At the morning session Professor Arnold Dresden, Assistant Secretary of the Society, gave a report on the business transacted by the Council at the April meeting in Chicago and by mail.

Vice-President Wedderburn presided, relieved by Ex-Presidents Fine and Veblen and Professor Alexander. At the beginning of the afternoon session addresses were presented, at the request of the program committee, by Mr. J. R. Carson and Dr. T. H. Gronwall on *The Heaviside operational calculus and its applications to electric circuit theory*. A number of engineers and physicists were present during this session by invitation, in addition to members of the Society.

Titles and abstracts of the papers presented at this meeting follow below. The papers of Dr. Castellani and Professor Hollcroft were read by title.

1. Professor Norbert Wiener: *The operational calculus.*

The author employs the method of separating the Fourier integral of a function into low and high frequency ranges to justify the asymptotic expansions derived by Oliver Heaviside by operational methods for the solution of the differential equations of the electric circuit.

2. Professor Norbert Wiener and Dr. Philip Franklin: *Analytic approximations to topological transformations.*

The leading theorem of this paper asserts the existence of an analytic transformation approximating to any degree of exactness any given one to one continuous transformation of the spherical surface into itself. The analytic transformation is obtained after an approximation with continuous partial derivatives is set up. This last construction depends on a dissection of the sphere and its transform into convex polygons which are nearly transforms of one another. All the vertices are triple, and we use affine transformations at these vertices. We then interpolate our transformation appropriately on the sides and in the interiors.

3. Dr. Philip Franklin: *Functions with assigned derivatives.*

In this paper it is shown that, given an infinite number of isolated points in the complex plane, with one branch cut drawn to infinity from each point, a function exists in the cut plane which is analytic at all points except those of the given set, and at these points possesses derivatives of all orders whose values may be arbitrarily assigned. The method of proof is an extension of that used by J. F. Ritt for a simpler, but related, problem. The function is also shown to exist when the further condition is imposed on it that, in a finite region not including any points on the branch cuts, it approximates uniformly to a given degree of exactness a given analytic function.

4. Professor Olive C. Hazlett: *On the types of division algebras.*

In this paper the author extends somewhat the results of a paper by Dickson in the TRANSACTIONS OF THIS SOCIETY for 1924 and one by Wedderburn in the TRANSACTIONS for 1921. It is proved that any quadratic division algebra of the type discussed by Cezioni in the RENDICONTI DI PALERMO for 1923 is a direct product of Dickson algebras.

5. Professor T. R. Hollcroft: *On conditions for self-dual curves.*

In accordance with the Lefschetz postulate of singularities for a given order $n \geqq 14$, self-dual plane curves exist only for a genus within and including the limits 0 and $2n-7$. This theorem results from Plücker's equations

and a theorem of Klein's: For an algebraic curve whose equation has real coefficients, a self-dual curve, and a self-dual curve only, may have all of its point and line singularities (1) real with real tangents and points, (2) imaginary. Six sets of equalities among the singularities of a curve are found all of which are necessary for the curve to be self-dual. Of these, three sets only are always sufficient. The remaining three are sufficient except for certain definite curve types, dual in pairs. For space self-dual curves, in general the plane projection curve is not self-dual, and conversely. Such curves whose plane projection curves are also self-dual exist only for $n \geq 8$ and for a very limited genus. There are eight equalities among the characteristics of space curves, four of which are both necessary and sufficient for the curve to be self-dual, and four necessary, but sufficient only with the exception of certain curve types.

6. Professor O. E. Glenn: *The invariant system of two associated bilinear connexes.*

Systems of invariants of connexes were studied by Clebsch and Gordan, who published, jointly, a paper on the theory in MATHEMATISCHE ANNALEN, vol. 1. The subject of complete systems remained stationary until 1916, when the present writer improved the algorism for simultaneous connexes (TRANSACTIONS OF THIS SOCIETY, vol. 17), and determined a complete system for a conic p_x^2 and a connex $a_x \alpha_u$. The present paper gives the determination of a fundamental system of two bilinear connexes $a_x \alpha_u$, $p_x \theta_u$. Subject to some reduction methods yet to be tried, the system contains, approximately, 130 invariant formations.

7. Dr. Maria Castellani: *Algebraic surfaces with reducible bitangent and osculating hyperplanar sections.*

The object of this note is to study, first, the algebraic surfaces in hyperspace which are cut by any bitangent hyperplane in reducible curves and, second, algebraic surfaces which are cut by any osculating hyperplane in reducible curves. Among the results is the following theorem: The algebraic irreducible surfaces $S_r (r > 5)$, not cones, which any bitangent hyperplane cuts in reducible curves are the ruled surfaces and the surfaces with hyperplanar sections of genus $p = 1$.

8. Professor J. H. M. Wedderburn: *The elementary divisors of a real symmetric matrix.*

This paper gives a short proof of the well known theorem that a real symmetric matrix has real roots and simple elementary divisors. The proof is based on the reduced characteristic equation introduced by Frobenius and on the fact that $a^2+b^2=0$ implies $a=b=0$ when a and b are real symmetric matrices.

9. Professor J. H. M. Wedderburn: *The absolute value of the product of two matrices.*

This paper appears in the July number of this BULLETIN.

10. Dr. T. H. Gronwall: *The algebraic structure of the formulas in plane trigonometry.*

The set of formulas connecting the sides and angles of a plane triangle is investigated from the point of view of Hilbert's theory of algebraic forms (MATHEMATISCHE ANNALEN, vol. 36). A linear basis of the set of formulas is constructed, the syzygies of various orders are set up, and finally the group of birational transformations leaving the formulas invariant is considered.

11. Dr. T. H. Gronwall: *The behavior at infinity of the gamma and associated functions.*

Regions are constructed in which the functions $\Gamma(z)$, $P(z)$, and $Q(z)$ tend uniformly to a limit as $|z|\to\infty$ (the limit being either zero or infinity) and asymptotic formulas are given for the roots of the equations $\Gamma(z)=a$, $P(z)=a$, $Q(z)=a$.

12. Professor Norbert Wiener: *A canonical form for biunivocal continuous sense-preserving transformations of a sphere.*

The author proves a theorem suggested by Professor Alexander, to the effect that it is possible to interpolate a continuous sequence of continuous biunivocal transformations between two biunivocal continuous sense-preserving transformations of a sphere into itself. In the course of this argument, these transformations are reduced to a canonical form.

13. Dr. Harry Levy: *Ricci's canonical congruences.*

Ricci has associated with a given congruence of curves in a Riemann space n—1 others which he calls the congruences *canonical* with respect to the given one. In this paper we obtain a geometrical interpretation of the canonical congruences dependent upon the parallelism of Levi-Civita; we derive the conditions that congruences be mutually canonical, and from them we deduce several theorems of geometric interest.

14. Mr. Orrin Frink: *Operations of boolean algebras.*

In this paper the problem of finding all operations of boolean algebras which have all the formal properties of a given boolean operation is solved by means of a theory of transformation. The author also finds all boolean operations which are unique, i. e., which share all their formal properties with no one other operation. In terms of such a unique triadic operation taken as a single fundamental operation a postulate system for boolean algebras is given. All boolean operations with idemfacient and nilfacient elements are determined. A method is given of deriving from a non-invariant relation between n elements an invariant relation between $n+1$ elements, with similar properties.

15. Mr. Alonzo Church: *Alternatives to Zermelo's assumption.* Preliminary report.

If we reject the axiom of choice as a principle of logic applicable to all classes, the following possibilities for the second ordinal class appear: (A) There exists an assignment of a unique fundamental series to every ordinal of the second kind in the second ordinal class. (B) There exists no assignment of a unique fundamental series to every ordinal of the second kind in the second ordinal class; but, given any ordinal, α, of the second ordinal class, there exists an assignment of a unique fundamental series to every ordinal of the second kind less than α. (C) There is an ordinal, α, of the second ordinal class, such that there exists no assignment of a unique fundamental series to every ordinal of the second kind less than α. These three possibilities may be regarded as postulates, determining three distinct kinds of second ordinal classes, and the consequences of each may be investigated on this basis, with an open mind about the possibility of obtaining

a contradiction from one or more of them. Any one of these postulates is found to be an effective instrument in obtaining theorems. On grounds of elegance, the theory resulting from (B) seems preferable, although it is in contradiction to the axiom of choice.

16. Professor Oswald Veblen and Dr. J. M. Thomas: *Projective normal coordinates for the geometry of paths.*

This paper introduces into the geometry of paths a system of coordinates which have the following properties: (1) they are independent of the particular affine connection associated with the paths and are therefore projective in character; (2) in terms of them the equations of the paths through the origin are linear in a specially chosen projective parameter; (3) they undergo a linear fractional transformation when the general coordinate system suffers an arbitrary analytic transformation. A projective normal tensor Q^i_{jkl} is defined in terms of them, and the Weyl projective curvature tensor is found to be $Q^i_{jkl}-Q^i_{jlk}$. An outline of this paper appeared in the PROCEEDINGS OF THE NATIONAL ACADEMY for April, 1925.

17. Dr. J. M. Thomas: *Conformal correspondence of Riemann spaces.*

In this paper is pointed out the existence of a set of quantities which are formed from the Christoffel symbols of the second kind and have the same value at corresponding points of two Riemann spaces mapped conformally on each other so that corresponding points have the same coordinates in both spaces. These quantities are useful in obtaining conformal invariants. The conformal curvature tensor of Weyl is obtained by expressing integrability conditions of their law of transformation under a change of coordinate system.

18. Dr. J. M. Thomas: *Asymmetric displacement of a vector.*

The changes in the components of a vector ξ^i by displacement from a point x^i to a nearby point x^i+dx^i of an n-dimensional manifold are assumed to be $-H^i_{jk}\xi^j\,dx^k$, where the quantities H have the same law of transformation as the Christoffel symbols of the second kind for a Riemann space, but are otherwise absolutely general. When the H's are changed, it is found that the directions of all displaced

vectors are preserved if, and only if, the changes in the symmetric and skew symmetric parts of H have the forms $\delta^i_j\varphi_k+\delta^i_k\varphi_j$ and $\delta^i_j\varphi_k-\delta^i_k\varphi_j$ respectively, φ_j denoting an arbitrary vector. The most general change of affine connection which preserves geodesics (cf. Weyl, GÖTTINGER NACHRICHTEN, 1921, p. 99) is brought about, therefore, by such a change in the H's. Tensors independent of φ_j are found, and a process for forming tensors of the same nature but of higher rank is given. A necessary and sufficient condition for the H's to assume the special form considered by Friedmann and Schouten (MATHEMATISCHE ZEITSCHRIFT, vol. 21, p. 18) is the existence of a coordinate system in which the original and displaced vectors have proportional components. The corresponding H's can be made symmetric by a change preserving displaced directions.

19. Dr. J. H. Taylor: *On the inverse problem of the calculus of variations.*

In this paper application of existing theory is made to the problem of determining when a given system of differential equations are such that they define the extremals for a definite integral, whose integrand is expressed in parametric form and which is homogeneous of degree one in the first derivatives. It is shown that if the "paths" of the geometry of paths minimize a definite integral of the above type its integrand is necessarily the square root of a quadratic form, that is, the paths then admit of being the geodesics of a Riemann space.

20. Dr. J. P. Ballantine: *On a certain functional equation.*

In taking a mean between two numbers x_1 and x_2, one has an equation of the form $p_1f(x_1)+p_2f(x_2)=(p_1+p_2)f(x_3)$. The mean based on the function f is called the f-mean. The problem is to find all functions f whose f-mean has the property that if x_1 and x_2 are each multiplied by the same number a, then the mean, x_3, is also multiplied by that same number. Additive and multiplicative constants in f have no effect on the mean, and, disregarding these, it is shown that x^n and $\log x$ are the only such functions. If a instead of being multiplied by x_1, x_2, and x_3 is added to them, the only functions are x and k^x, where k is an arbitrary constant.

ARNOLD DRESDEN,
Assistant Secretary.

THE FORTY-SIXTH REGULAR MEETING OF THE SAN FRANCISCO SECTION

The forty-sixth regular meeting of the San Francisco Section of the Society was held at the University of Oregon on Friday, June 19, 1925. Professor E. R. Hedrick presided and Professor W. E. Milne acted as Secretary. The total attendance was eighteen, including the following fourteen members of the Society:

Bernstein, A. F. Carpenter, De Cou, Dederick, Eells, E. R. Hedrick, Kent, Lehmer, W. E. Milne, Neikirk, Moritz, Robertson, Smail, Winger.

It was decided to hold the next Northwest meeting of the Section at the University of Washington on a date to be determined later.

The visiting members and their families were generously entertained by the University of Oregon members on the day of the meeting and on the day following.

Titles and abstracts of papers read at the meeting follow. The papers of Bell, Cajori, Cramlet, and Stager were read by title.

1. Professor E. T. Bell: *On the representations of integers as sums or differences of two positive integer cubes.*

This paper appeared in the July number of this BULLETIN.

2. Professor E. T. Bell: *Generalizations of the eight-square and similar identities.*

A generalization for the eight-square identity similar to that of Lagrange for the four-square is given; and it is shown that the corresponding norm theorems for algebraic numbers can be similarly generalized. This paper will appear in the ANNALS OF MATHEMATICS.

3. Professor B. A. Bernstein: *Sets of postulates for the logic of propositions.*

Defining, with Schröder, the logic of propositions as a two-element logic of classes, the author obtains four extremely simple sets of postulates for this logic, the number

of the postulates being respectively 5, 4, 3, 1. The complete existential theory is established for each set.

4. Professor Florian Cajori: *Newton's indebtedness to Descartes.*

Ordinarily, we think only of the long struggle for supremacy between the Newtonian and Cartesian explanations of the mechanics of the solar system, and we overlook Newton's indebtedness to Descartes, in such cases as the following: (1) Descartes, rather than Euclid, initiated the young Newton into geometry, (2) Descartes' algebra, not Oughtred's or Harriot's, familiarized Newton with the modern exponential notation for positive integral exponents which he later generalized by the introduction of negative and fractional exponents, (3) Newton studied Descartes' rule of signs for equations, and gave it a precise statement, (4) he acknowledged the Cartesian optical theory to be a "good step," (5) he referred appreciatively to Descartes' theory of vision, (6) he used Descartes' (and Snell's) law of refraction in the explanation of total reflection, (7) according to Conduit, it was Descartes' physical theory of color that started Newton's experiments with prisms.

5. Professor A. F. Carpenter: *Self-correspondence of the complex developables of a ruled surface under a point-line transformation of space.*

By means of a transformation of the points of space into the lines of a linear congruence, the developables whose cuspidal edges are the two branches of the complex curve of a ruled surface are shown to be self-corresponding in such a way that to the points of any curve C lying upon one of these developables correspond the lines of that developable which are intersected by C.

6. Professor D. N. Lehmer: *Note on the construction of tables of linear forms belonging to a given quadratic residue.*

This paper appears in full in the present number of this BULLETIN.

7. Mr. C. M. Cramlet: *Some determinant theorems as tensor equations.*

The h^2 differential equations $a_{ir}(\partial\varphi/\partial a_{ij}) = \varphi \cdot \delta_r^j$ have $\varphi = Aa$ as a general solution, where a is the determinant

of the a_{ij}'s and A is a constant of integration. The most general operations with determinants which merely introduce a factor are found from the condition that these equations be invariant. The determination of the constant of integration under these transformations gives the multiplication theorems.

Using a tensor defined by Murnaghan in a note in the AMERICAN MATHEMATICAL MONTHLY, vol. 32, a list of general determinant theorems is given in tensor notation. It is shown that a determinant of a tensor of second rank is a tensor with h^{2n} components of $+a$'s, $-a$'s, and 0's. Knowing this, the relative tensor, discussed by Veblen and Thomas, TRANSACTIONS OF THIS SOCIETY, vol. 26, ceases to be an extension. Similar investigations will be made with ϱ-way determinants.

8. Professor R. E. Moritz: *A new generalization of Wilson's theorem.*

In volume 31 of the QUARTERLY JOURNAL OF MATHEMATICS, Glaisher proved that the sum of the products of the first $n-1$ integers taken $p-1$ together, p being any prime not greater than n, is congruent to $-r$, mod p, where r is the integral part of the quotient obtained on dividing n by p. If $n = p$ we have Wilson's theorem. The author shows that this theorem may be extended, with certain restrictions, to any n successive integers.

9. Professor R. E. Moritz: *A theorem in number congruences.*

The author proves that if n is any prime number greater than 3, the combinations of all numbers of the form $kn-1$ taken $n-1$ together leave the same remainder when divided by n^3, that is to say

$$C_{n-1}^{kn-1} - C_{n-1}^{ln-1} \equiv 0, \pmod{n^3}.$$

When $l = 1$, we have

$$C_{n-1}^{kn-1} \equiv 1, \pmod{n^3}.$$

The special case of our corollary for $k = 2$ has been previously proved by Wolstenholme, QUARTERLY JOURNAL OF MATHEMATICS, volume 5.

10. Dr. H. P. Robertson: *On certain solutions of Einstein's cosmological equations.* Second paper.

The author here considers manifolds whose line elements are of the form

$$ds^2 = f(x, y, z, t)\{dx^2 + dy^2 + dz^2\} - g(x, y, z, t)dt^2,$$

i. e., orthogonal four-spaces containing a family of three-spaces, $t =$ constant. The cosmological equations for the case of a dynamical field of this type are immediately reduced to six second order equations in one dependent and four independent variables. These equations contain six arbitrary functions (arising through the integration of third order equations) which are determined by the conditions of integrability. All manifolds of this type are found; interpreted as relativistic space-times they represent all dynamical worlds which admit of orthogonal coordinates in which the velocity of light is independent of direction at a point. The solutions yield geometric rather than physical interpretations.

11. Dr. H. W. Stager: *A factor table of the second and succeeding ten millions.*

In this paper the author describes the method to be employed in constructing a factor table to the ultimate limit of 100,000,000. Following out the general method used in constructing his *Sylow Factor Table,* the table will first be extended to 50,000,000. The use of an automatic device and a special form of rubber type in making the entries assures almost absolute accuracy. These tables will be of the same general form as Lehmer's well known *Factor Table,* but will only show the least divisor of numbers not divisible by 2, 3, 5, 7, or 11. The omission of numbers divisible by 11 will reduce the size of the volume for each 10,000,000 approximately one-eleventh, without materially increasing the labor of finding the factors of a number.

12. Professor R. M. Winger: *The ternary Hesse group and its invariants.*

The Hesse collineation group has received attention at the hands of several writers, including Jordan, Witting, Muth, Maschke, Newson, Burnside, Steinitz, and Blichfeldt. The present author makes a systematic analysis of the group on the basis of its generating transformations and the theory of collineations, discussing in particular those properties of the invariant configuration and curves which follow most directly from the properties of the group.

B. A. BERNSTEIN,
Secretary of the Section.

ON THE NUMBER OF ELEMENTS OF A GROUP WHICH HAVE A POWER IN A GIVEN CONJUGATE SET*

BY LOUIS WEISNER

1. *Introduction.* A fundamental theorem on abstract groups is Frobenius' theorem: The number of elements in a group of order g whose nth powers belong to a given conjugate set is zero or a multiple of the greatest common divisor of g and n. In this paper, I will prove the following theorems, which are also concerned with the number of elements having a power in a given conjugate set.

THEOREM 1. *The number of elements of a group whose nth powers are in a given conjugate set is either zero, or a multiple of the number of elements in the conjugate set.*

THEOREM 2. *In a group of order g, the number of elements which have a power in a given conjugate set of elements of order n is a multiple of the greatest divisor of g that is prime to n.*

An interesting deduction from Theorem 2 is the following theorem.

THEOREM 3. *In a group of order g, the number of elements whose orders are multiples of n is either zero, or a multiple of the greatest divisor of g that is prime to n.*

2. *Proof of Theorem* 1. Let $t_1, t_2, \cdots, t_x$ be the elements of a group G which satisfy the equation $t^n = s_1$, and let the conjugates of s_1 under G be $s_1, s_2, \cdots, s_m$. There exist elements $u_1, u_2, \cdots, u_m$ in G such that

$$u_i^{-1} s u_i = s_i, \qquad (i = 1, 2, \cdots, m).$$

Since $t_a^n = s_1$,

$$(u_i^{-1} t_a u_i)^n = u_i^{-1} s u = s_i.$$

* Presented to the Society, February 28, 1925.

Hence there are exactly x elements of G whose nth powers are s_i. If $u_i^{-1}t_a u_i = u_k^{-1}t_b u_k$, then, raising both members to the nth power, we have

$$u_i^{-1}s u_i = u_k^{-1}s u_k,$$

or $s_i = s_k$, whence $i = k$. It follows that $t_a = t_b$, whence $a = b$.

The distinct elements of G whose nth powers are conjugate to s_1 are therefore

$$u_i^{-1}t_a u_i, \qquad \begin{pmatrix} a = 1, 2, \cdots, x \\ i = 1, 2, \cdots, m \end{pmatrix},$$

and their number is mx.

3. *Proof of Theorem* 2. Suppose, first, that the conjugate set consists of only one element s, which is therefore invariant under the group G. Let k be the greatest divisor of g that is prime to n; and let $t^a = s$.

Case 1: *a prime to n*. If $aa' \equiv 1 \pmod{n}$, then $t = s^{a'}$. An element u of G of order m prime to n, being commutative with s, is commutative with t. Hence

$$(tu)^{xm} = s^{a'xm}u^{xm} = s,$$

where $xa'm \equiv 1 \pmod{n}$. *Hence s is a power of tu.*

By Frobenius' theorem, G contains λk (λ integral) elements whose orders divide k. Denote these elements by $u_1, \cdots, u_{\lambda k}$. We have just proved that the only elements of G satisfying the equation $t^a = s$, where a assumes all values prime n, are

$$(1) \qquad s^{c_\mu}u_z, \qquad \begin{pmatrix} \mu = 1, \cdots, \varphi(n) \\ z = 1, \cdots, k \end{pmatrix},$$

where $c_1, \cdots, c_{\varphi(n)}$ are the integers not greater than n and prime to n. These elements are distinct;* hence their number is a multiple of k.

Case 2: *a not prime to n*. Let $a = n_1 b$, where b is the greatest divisor of a that is prime to n. We may write $t = t_1 t_2$,† where t_1 and t_2 are powers of t, and the order of t_2 is the greatest divisor of the order of t that is prime to n, while the order of t_1 is a multiple n_2 of n which is not divisible by a number prime to n (except

* W. Burnside, *Theory of Groups*, 1911, § 16.

† Burnside, loc. cit.

unity). If the order of t_2 is c, then the order of t is n_2c. Since $s = t^{n_1b} = t_1^{n_1b}t_2^{n_1b}$, we may write

$$s^n = t_1^{nn_1b}t_2^{nn_1b} = 1, \tag{2}$$

whence

$$t_1^{nn_1b} = t_2^{nn_1b} = 1.$$

Hence c is a divisor of nn_1b and therefore of b. It follows from (2) that $s = t_1^{n_1b}$. If u, of order m prime to n, is commutative with t, and if $m'm \equiv 1 \pmod{n_2}$, then

$$(t_1u)^{m'mn_1b} = t_1^{m'mn_1b}u^{m'mn_1b} = t_1^{n_1b} = s. \tag{3}$$

Hence s is a power of t_1u.

Let n_3k_1 be the order of the normaliser N of t in G and let $g/n_3k_1 = n_4k_2$, where k_1 and k_2 are the greatest divisors of n_3k_1 and n_4k_2 respectively that are prime to n and hence to n_2. The number of elements of N whose orders are prime to n is of the form αk_1; denote these elements by

$$u_1, \cdots, u_{\alpha k_1}. \tag{4}$$

It follows from (3) that s is a power of

$$t_1u_1, \cdots, t_1u_{\alpha k_1}. \tag{5}$$

It is noteworthy that t_2 is in (4) and hence $t = t_1t_2$ is in (5). Let $t_1' = w^{-1}t_1w$ be a conjugate of t_1. Since s is invariant under G,

$$t_1'^{n_1b} = w^{-1}t_1^{n_1b}w = w^{-1}sw = s.$$

Now there are exactly αk_1 elements in G whose orders are prime to n and which are commutative with t_1'. Denoting these by $u_1', \cdots, u_{\alpha k_1}'$, it follows that s is a power of

$$t_1'u_1', \cdots, t_1'u_{\alpha k_1}'. \tag{6}$$

Moreover, no element in (6) is equal to an element in (5).* There being n_4k_2 conjugates of t_1, we obtain $\alpha n_4k_1k_2$ elements of which s is a power. Observing that $k = k_1k_2$ is the greatest divisor of g prime to n, it follows that t_1 and its conjugates give rise in the manner described above to a

* Burnside, loc. cit.

multiple of k elements of which s is a power. These elements are evidently distinct from those obtained under Case 1.

If s is a power of τ and τ is not one of the elements already obtained, let $\tau = \tau_1\tau_2$, where the order of τ_2 is the greatest divisor of the order of τ that is prime to n. Then τ_1 and its conjugates give rise to a multiple of k elements of which s is a power. Let $\tau_1 v$ be one of these elements, and, if possible, let it be equal to an element in (5), say $\tau_1 v = t_1 u = v$. Since the order of u and the order of v are both equal to the greatest divisor of the order of w that is prime to n, we must have $\tau_1 = t_1$, $v = u$. This is not the case, and hence τ_1 and its conjugates give rise to a multiple of k new elements of which s is a power.

The theorem now follows under the assumption that s is invariant under G.

Suppose next that the conjugate set consists of powers of s, so that (s) is invariant under G. Denote the conjugates of s by

$$s^{c_1}, \cdots, s^{c_r}, \qquad (c_1 = 1), (r > 1). \tag{7}$$

Let H, of order h, be the normaliser of s in G; and let k be the greatest divisor of h that is prime to n. The number of elements in H of which s is a power is a multiple of k, which we denote by λk. Since $c_1, \cdots, c_r$ are prime to n, $s^{c_1}, \cdots, s^{c_r}$ are powers of these same λk elements. Hence H contains exactly λk elements which have a power in (7). If t has a power in (7), so has t^{c_i} $(i = 1, 2, \cdots, r)$. Hence the number of elements of G which have powers in (7) is a multiple of r. The order of G is $g = rh$; for G/H is simply isomorphic with the group obtained by establishing an isomorphism of s with $s^{c_1}, \cdots, s^{c_r}$, and is of order r. The number of elements of G which have powers in (7) is a multiple of r and a multiple of k and is therefore a multiple of the greatest divisor of g that is prime to n.

Finally, suppose the conjugate set of elements does not consist of the powers of one of them. The elements in the conjugate set may be separated into subsets

$$(8_1) \qquad s_1^{c_1}, \cdots, s_1^{c_r}, \qquad (c_1 = 1),$$
$$(8_2) \qquad s_2^{c_1}, \cdots, s_2^{c_r}, \qquad (r \geqq 1),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(8_m) \qquad s_m^{c_1}, \cdots, s_m^{c_r}, \qquad (m > 1),$$

such that any two elements in the same subset are powers of each other, whereas no element is a power of an element in another subset.

Let H_i, of order h, be the normaliser of (s_i), $(i = 1, \cdots, m)$. An element of G which has a power in (8_i) is commutative with the elements of (8_i) and hence is in H_i. Therefore H_i contains all the elements of G which have powers in (8_i). Since the elements of (8_i) form a complete set of conjugates under H_i, the number of elements of G which have powers in (8_i) is λk, where λ is an integer, and k is the greatest divisor of h that is prime to n. If t has a power in (8_i), t cannot have a power in (8_j), $(i \neq j)$. For if $t^x = s_i^c$ and $t^y = s_j^{c'}$, then t^x and t^y are of the same order n, so that each is a power of the other, whence $i = j$. It follows that the number of elements of G which have a power in (8) is $\lambda kg/h$, and this number is evidently a multiple of the greatest divisor of g that is prime to n.

4. *Proof of Theorem* 3. If the group G contains an element of order n, separate the elements of order n into complete sets of conjugates, which we denote by $C_1, \cdots, C_x$. Of these, we select a subset $C_1, \cdots, C_y$, such that no element in C_i has a power in C_j $(i \neq j)$, $(i, j = 1, \cdots, y)$. Every element of G whose order is a multiple of n has a power in one and only one of the sets $C_1, \cdots, C_y$; and the order of every element which has a power in one of these sets is a multiple of n. Since the number of elements of G which have a power in C_i is a multiple of the greatest divisor of g that is prime to n, it follows that the number of elements of G whose orders are multiples of n is a multiple of the greatest divisor of g that is prime to n.

UNIVERSITY OF ROCHESTER

NOTE ON THE CONSTRUCTION OF TABLES OF LINEAR FORMS*

BY D. N. LEHMER

Certain theorems concerning the linear forms that go with a given quadratic residue seem not to have been noticed, or if noticed, not to have been appreciated by those who have constructed tables of divisors of quadratic forms. In connection with work on factor stencils, the author has found it necessary to extend these tables of linear forms, and the construction and checking has been much facilitated and rendered more accurate by the simple observation that if a and b are two entries in such a table so also is the product ab. In particular, the powers of any entry are also in the list of entries. This indicates a very rapid method of computation that is well adapted to a computing machine. If R is of the form $4n+1$, then in the table for R, if an entry a appears so also will $-a$. This furnishes an important check on the accuracy of the work. The numerous errors in Legendre's tables could not have escaped detection if this check had been employed.

The table for $-R$ may be easily written down without computation from the table for $+R$ and conversely, when it is observed that entries of the form $4n+1$ appear in the same place in both tables while entries of the form $4n-1$ which appear in one table must not appear in the other.

Availing himself of these theorems and checks, the author has already extended the tables of forms as far as $+300$ and -300. The tables for $+58$ and -58 are subjoined to indicate the form used and to illustrate the method of construction and the checks. Thus the first table shows that the forms for $+58$ are $232n+1$, 3, 7, 9, 11, 19, 21, 23, 25, 27, 33, $\cdots$, these entries being indicated by 1's. Numbers not prime to the modulus 58 are indicated by

* Presented to the Society, San Francisco Section, June 19, 1925.

circles. Since $(58/3) = 1$, we can start with the entry 3 in this table and enter the powers of 3, casting out multiples of 232 as they arise. This process gives all the entries for this table. The entries may appear in several cycles, of course, instead of one as in this case.

The table for $-R$ is now built from this by setting over into it the entries of the form $4n+1$ as they stand, such as 1, 9, 21, 25, 33, 37, $\cdots$, and then filling in the blanks in the first table which correspond to numbers of the form $4n-1$, such as 15, 31, 35, 39, 47, $\cdots$, avoiding the numbers marked with circles. It is then carefully verified that the table for $+58$ reads backwards and forwards the same, while in the table for -58 the blanks at one end of the table match with the 1's at the other. The theorems upon which these checks are based are easily deduced from Jacobi's extension of Legendre's Law of Reciprocity.

$R = +58$

	1	3	5	7	9
0	1	1		1	1
1	1				1
2	1	1	1	1	○
3		1		1	
4		1			1
5				1	
6	1	1	1		1
7	1		1	1	
8	1		1	○	
9					1
10	1	1			
1	1				
2	1				1
3	1	1			
4			○	1	
5	1		1	1	
6	1	1		1	1
7	1		1		
8		1			1
9			1		1
20		○	1	1	1
1	1	1			
2	1	1	1		1
3	1				

$R = -58$

	1	3	5	7	9
0	1				1
1			1		
2	1		1		○
3	1	1	1	1	1
4				1	1
5	1		1	1	1
6	1		1	1	1
7				1	1
8	1	1	1	○	
9	1		1		
10	1			1	
1			1		1
2	1	1		1	1
3		1	1		1
4		1	○		
5				1	1
6	1				1
7					1
8				1	1
9	1				
20		○	1		1
1		1	1		1
2	1		1	1	1
3					

THE UNIVERSITY OF CALIFORNIA

CRITERIA THAT ANY NUMBER OF REAL POINTS IN n-SPACE SHALL LIE IN AN $(n-k)$-SPACE

BY H. S. UHLER

The object of the present paper is to establish an algebraic identity from which may be deduced necessary and sufficient conditions that any large number of real points in n-dimensional linear space shall lie in a linear $(n-k)$-space.

Let the following matrix, in which the number of columns is m and the number of rows is $n+1\,[m \geqq (n+1)]$, be compounded with its conjugate:

$$\begin{matrix} 1 & 1 & \cdot & \cdot & \cdot & 1 \\ x_{1,1} & x_{2,1} & \cdot & \cdot & \cdot & x_{m,1} \\ x_{1,2} & x_{2,2} & \cdot & \cdot & \cdot & x_{m,2} \\ \cdot & \cdot & & \cdot & & \cdot \\ x_{1,n} & x_{2,n} & \cdot & \cdot & \cdot & x_{m,n}. \end{matrix}$$

The determinant of the resulting symmetric square array is

$$\begin{vmatrix} m & \Sigma x_{i,1} & \Sigma x_{i,2} & \cdot & \cdot & \cdot & \Sigma x_{i,n} \\ \Sigma x_{i,1} & \Sigma x_{i,1}x_{i,1} & \Sigma x_{i,1}x_{i,2} & \cdot & \cdot & \cdot & \Sigma x_{i,1}x_{i,n} \\ \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \Sigma x_{i,n} & \Sigma x_{i,n}x_{i,1} & \Sigma x_{i,n}x_{i,2} & \cdot & \cdot & \cdot & \Sigma x_{i,n}x_{i,n} \end{vmatrix} \equiv \Delta;$$

$$(i = 1, 2, 3, \cdots, m).$$

Multiply all of the rows of Δ except the top row by m, compensate by prefixing m^{-n}, and remove the factor m now common to the constituents of the first column to get

$$\Delta = m^{1-n} \begin{vmatrix} 1 & \Sigma x_{i,1} & \Sigma x_{i,2} & \cdots & \Sigma x_{i,n} \\ \Sigma x_{i,1} & m\Sigma x_{i,1}x_{i,1} & m\Sigma x_{i,1}x_{i,2} & \cdots & m\Sigma x_{i,1}x_{i,n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \Sigma x_{i,n} & m\Sigma x_{i,n}x_{i,1} & m\Sigma x_{i,n}x_{i,2} & \cdots & m\Sigma x_{i,n}x_{i,n} \end{vmatrix};$$

$$(i = 1, 2, 3, \cdots, m).$$

Next subtract $\sum_{i=1}^{i=m} x_{i,k}$ times the first column from the $(k+1)$th column, $(k = 1, 2, 3, \cdots, n)$, in order to reduce to zero all the constituents of the top row, except the leading constituent, and to find $\Delta = U_n/m^{n-1}$, where

$$U_n \equiv \begin{vmatrix} \sigma_{1,1} & \sigma_{1,2} & \cdot & \cdot & \cdot & \sigma_{1,n} \\ \sigma_{2,1} & \sigma_{2,2} & \cdot & \cdot & \cdot & \sigma_{2,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \sigma_{n,1} & \sigma_{n,2} & \cdot & \cdot & \cdot & \sigma_{n,n} \end{vmatrix}$$

and

$$\sigma_{p,q} \equiv m \sum_{i=1}^{i=m} x_{i,p} x_{i,q} - \left(\sum_{i=1}^{i=m} x_{i,p} \right) \left(\sum_{i=1}^{i=m} x_{i,q} \right)$$

$$= \sum_{i=j+1}^{i=m} \sum_{j=1}^{j=m-1} [(x_{i,p} - x_{j,p})(x_{i,q} - x_{j,q})] = \sigma_{q,p}.$$

Now the determinant Δ produced by compounding the matrices specified above is known to equal the sum of the squares of all the ν determinants of order $n+1$ that can be formed from the columns of the original matrix, where

$$\nu \equiv \binom{m}{n+1}.$$

Let any one of these determinants be denoted by D_r; then the required algebraic identity is

$$U_n = m^{n-1} \sum_{r=1}^{r=\nu} (D_r^2). \tag{1}$$

Thus far no special meaning has been assigned to the x's; they may represent complex quantities, etc.

To obtain the criteria contemplated advantage will be taken of the fact that D_r is squared in identity (1) so that if the x's are real numbers D_r^2 will be incapable of becoming negative. Accordingly let the rectangular coordinates of a system of real points in n-dimensional flat space be

$$(x_{i,1}, x_{i,2}, \cdots, x_{i,n}); \; i = 1, 2, 3, \cdots, m; \; m \geqq (n+1).$$

Also let S_t symbolize a linear space of t dimensions, a t-flat.

Now the vanishing of $\sum(D_r^2)$ is a necessary and sufficient condition that the m given real points shall lie in the same S_{n-1}, hence, by formula (1), *a necessary and sufficient condition that any number* $m\,[\geqq(n+1)]$ *of real points in* S_n *shall lie in the same* S_{n-1} *is the vanishing of* U_n.

When $m = n+1$, $\nu = 1$ so that there is only one D_r in $\sum(D_r^2)$. This D_r represents $n!$ times the content of the hyper-figure or simplex having the $n+1$ given points as vertices.* Hence, for $m>(n+1)$, $\sum(D_r^2)$ is proportional to the sum of the squares of the contents of all the simplexes that can be formed from the m points taken $n+1$ at a time as vertices of each geometric figure. Accordingly the above italicized statement may also be interpreted as meaning that the contents of all the simplexes involved vanish.

Keeping $m=n+1$, and giving n successively the values $1, 2, 3, 4, \cdots, n$, we may derive from the identity (1) the following expressions for the respective magnitudes of the length of a segment in S_1, the area of a triangle in S_2, the volume of a tetrahedron in S_3, the hyper-volume of a pentahedroid in $S_4, \cdots$, the content of a simplex in S_n:

$$\sigma_{1,1}^{1/2}, \quad \frac{|\sigma_{1,1},\ \sigma_{2,2}|^{1/2}}{2\sqrt{3}}, \quad \frac{|\sigma_{1,1},\ \sigma_{2,2},\ \sigma_{3,3}|^{1/2}}{24},$$

$$\frac{|\sigma_{1,1},\ \sigma_{2,2},\ \sigma_{3,3},\ \sigma_{4,4}|^{1/2}}{120\sqrt{5}}, \cdots, \frac{|\sigma_{1,1},\ \sigma_{2,2}, \cdots, \sigma_{n,n}|^{1/2}}{n!\,(n+1)^{(n-1)/2}}.$$

The extension of the above italicized statement from S_{n-1} to S_{n-k} is an immediate consequence of the well known properties of orthogonal projections of linear spaces. The fundamental idea is that identity (1) holds for a smaller number of coordinates than n and hence it may be applied to the orthogonal projections of the m given points upon *all* of the

$$\binom{n}{n-k+1}$$

* P. H. Schoute, *Mehrdimensionale Geometrie*, Part 2, §§ 36, 37.

coordinate-S_{n-k+1}'s. In other words the original matrix is to be replaced by

$$\binom{n}{n-k+1}$$

matrices having the same top row of m 1's while the remaining rows are composed of $n-k+1$ of the original rows of x's. There will now be

$$\binom{n}{n-k+1}$$

new systems of points,—one in each coördinate-S_{n-k+1},—to all of which the above italicized test must be applied. The orders of the U_n's and D_r's of formula (1) will be $n-k+1$ and $n-k+2$ respectively. Without further comment it should be perfectly clear that *necessary and sufficient conditions that any number of real points in n-dimensional flat space shall lie in an $(n-k)$-dimensional flat space are that all the*

$$\binom{n}{n-k+1}$$

determinants U of order $n-k+1$ in the σ's shall vanish while one, at least, of the determinants U of order $n-k$ shall be finite.

The last sentence may be stated in terms of the rank of the U of order n.* Incidentally the writer has found it possible to express the general criteria analytically in terms of only two determinants involving polynomial constituents composed of the σ's.

YALE UNIVERSITY

* G. Kowalewski, *Determinantentheorie*, § 52.

CHARACTERISTIC PARAMETER VALUES FOR AN INTEGRAL EQUATION*

BY W. A. HURWITZ

1. *Introduction*; *Hermitian Kernel.* It is well known, though seldom explicitly mentioned, that the properties of real symmetric kernels hold, with only trivial alterations in the usual proofs, for complex Hermitian kernels. It is the purpose of this note to show that kernels of certain other kinds have analogous properties, with the role of the real axis as bearer of the characteristic parameter values taken by any other straight line in the plane.

All functions of x, y, s which appear will be understood to be complex functions defined for real values of each of the variables in the closed interval (a, b); all integrals will be taken between the limits a, b, which will not be written. We call λ a *characteristic parameter value* (hereafter abbreviated cpv) for the kernel $K(x, y)$ if there exist non-trivial solutions of the equations

$$u(x) = \lambda \int K(x, s)u(s)ds, \tag{1}$$

$$v(y) = \lambda \int v(s)K(s, y)ds. \tag{2}$$

The cpv's and the corresponding solutions of (1) (normalized and orthogonalized), arranged in the usual order, will sometimes be denoted by λ_p, $\varphi_p(x)$ $[p = 1, 2, \cdots]$. A cpv is a pole (for at least some values of x, y) of the resolvent function $K(x, y; \lambda)$, which is elsewhere analytic in λ. The resolvent satisfies the equation

$$K(x, y; \lambda) - K(x, y; \mu) = (\lambda - \mu) \int K(x, s; \lambda) K(s, y; \mu)ds \tag{3}$$

whenever neither λ nor μ is a cpv. Since $K(x, y; 0) = K(x, y)$, (3) reduces for $\lambda = 0$ and for $\mu = 0$ to the simpler resolvent formulas used in the process of solving the Fredholm equation.

* Presented to the Society, December 30, 1924.

A continuous* kernel $K(x, y)$ is *closed* if the equation $\int K(x, s)u(s)ds = 0$ is satisfied by no continuous function except $u(x) = 0$.

The properties of Hermitian kernels with which we are particularly concerned may be stated as follows:

If $K(x, y)$ is continuous and not identically zero, and† $\overline{K}(y, x) = K(x, y)$, *then* (I) *every cpv is real*; (II) *every pole of the resolvent is of the first order*; (III) *there is at least one cpv*; *if K is closed, the set of cpv's is infinite*; (IV) *if for a cpv λ, $\varphi(x)$ is a solution of* (1), *then $\overline{\varphi}(y)$ is a solution of* (2); (V) *the series*‡ $\sum \varphi_p(x)\overline{\varphi}_p(y)/\lambda_p^m$ *converges uniformly in x, y and represents $K_m(x, y)$ for $m \geqq 2$; it represents $K(x, y)$ for $m = 1$ provided it converges uniformly*; (VI) *any function $f(x)$ expressible in the form $\int K(x, s)u(s)ds$, where $u(x)$ is continuous, can be expanded in the uniformly convergent series $f(x) = \sum \varphi_p(x) \int \overline{\varphi}_p(s) f(s)ds$.*

2. *Straight Line through Origin.* Suppose that instead of the Hermitian condition we have $\overline{K}(y, x) = \zeta K(x, y)$, where K is not identically zero; then

$$\overline{K}(x, y) = \zeta K(y, x) = \zeta\overline{\zeta}\overline{K}(x, y),$$

so that $|\zeta| = 1$. Denote by $\sqrt{\zeta}$ either square root of ζ, and let $L(x, y) = \sqrt{\zeta}\, K(x, y)$. If λ, μ denote corresponding cpv's for K, L, then $\lambda = \mu\sqrt{\zeta}$. Since

$$\overline{L}(y, x) = \frac{1}{\sqrt{\zeta}}\,\overline{K}(y, x) = \sqrt{\zeta}\, K(x, y) = L(x, y),$$

L is Hermitian and μ is real. Thus λ lies on a line through the origin and the point $\sqrt{\zeta}$. The other properties are readily verified, and we have the following result.

---

* The assumption of continuity is made for simplicity; the usual extensions, with the use of Lebesgue integrals, are possible.

† A bar over a symbol for a *variable* denotes the conjugate of that variable; a bar over a symbol for a *function* denotes the conjugate of the whole expression represented by the function; for instance, $\overline{K}(x, y; \lambda)$ is an abbreviation for $\overline{K(x, y; \bar{\lambda})}$.

‡ If the number of cpv's is finite, the series terminates, and the "convergence" is uniform in x, y for any value of m.

If $K(x, y)$ is continuous and not identically zero, and $\bar{K}(y, x) = \zeta K(x, y)$, then $|\zeta| = 1$; (I) *every cpv is on the line through the origin and $\sqrt{\zeta}$;* (II–VI) *the other properties stated in* § 1 *hold without change.*

A case of especial interest is that of the skew-Hermitian kernel: $\bar{K}(y, x) = -K(x, y)$; here $\zeta = -1$ and the cpv's are pure imaginary. Current treatises* prove the results of § 1 only for real symmetric kernels, and are therefore able to deduce the corresponding facts for real skew-symmetric kernels only by repeating the proofs throughout; whereas the procedure here given makes the second theory essentially a mere special case of the first.

3. *Special Straight Line not through Origin.* Now suppose that K is not identically zero and satisfies one of the relations†

$$(4) \qquad K(x, y) + \bar{K}(y, x) = 2\int K(x, s)\bar{K}(y, s)ds,$$

$$(5) \qquad K(x, y) + \bar{K}(y, x) = 2\int \bar{K}(s, x)K(s, y)ds;$$

for definiteness, say (4). By (3), if $K(x, y; 2)$ exists,

$$K(x, y; 2) = K(x, y) + 2\int K(x, s)K(s, y; 2)ds;$$

and, conversely, this relation is satisfied only by $K(x, y; 2)$; hence (4) is equivalent to

$$(6) \qquad K(x, y; 2) = -\bar{K}(y, x),$$

and (5) holds also. Since K has at most a countably infinite set of cpv's, there is on the line $R(\lambda) = 1$ a point $\lambda = \lambda_0$ which is not a cpv; thus

$$(7) \qquad K(x, y; \lambda_0) = K(x, y) + \lambda_0\int K(x, s)K(s, y; \lambda_0)ds$$

* Lalesco, *Introduction à la Théorie des Équations Intégrales*, pp. 64, 73; Vivanti, *Elementi della Teoria delle Equazione Integrali Lineari*, pp. 207, 241.

† I omit the simple heuristic steps by which one is led to these equations as the natural generalization of the Hermitian condition.

and

$$(8)\quad K(x,y;\lambda_0) = K(x,y;2) + (\lambda_0-2)\int K(x,s;\lambda_0)K(s,y;2)ds.$$

From (8) we deduce, by interchanging x, y and taking conjugates,

$$(9)\quad \bar{K}(y,x;\lambda_0) = \bar{K}(y,x;2) + (\bar{\lambda}_0-2)\int \bar{K}(s,x;2)\bar{K}(y,s;\lambda_0)ds,$$

or, by (6) and the relation $\lambda_0+\bar{\lambda}_0 = 2$,

$$(10)\quad \bar{K}(y,x;\lambda_0) = -K(x,y)+\lambda_0\int K(x,s)\bar{K}(y,s;\lambda_0)ds.$$

Call $M(x,y) = K(x,y;\lambda_0)+\bar{K}(y,x;\lambda_0)$. By (7) and (10),

$$M(x,y) = \lambda_0\int K(x,s)M(s,y)ds,$$

so that $M(x,y) = 0$ and

$$(11)\quad \bar{K}(y,x;\lambda_0) = -K(x,y;\lambda_0).$$

If we write $N(x,y) = -K(x,y;\lambda_0)$, and call $N(x,y;\mu)$ the resolvent to $N(x,y)$, then $N(x,y;\lambda-\lambda_0) = K(x,y;\lambda)$; also

$$(12)\quad K(x,y) = N(x,y)-\lambda_0\int K(x,s)N(s,y)ds,$$
$$= N(x,y)-\lambda_0\int N(x,s)K(s,y)ds.$$

By (11), $\bar{N}(y,x) = -N(x,y)$; and by § 2, with $\zeta = -1$, any cpv μ for N is a pure imaginary: $R(\mu) = 0$. Thus a cpv λ for K must satisfy $R(\lambda-\lambda_0) = 0$ or $R(\lambda) = 1$.

This condition replaces (I) of § 1. The other conditions may also be carried over; only (V), (VI), and the second part of (III) require any special consideration, which we now proceed to give.

In view of (12), it is readily seen that the equations

$$(13)\quad f(x) = \int K(x,s)u(s)ds, \qquad f(x) = \int N(x,s)v(s)ds$$

are equivalent if $v(x) = u(x)-\lambda_0 f(x)$; hence the hypothesis concerning f in (V), involving K, may be stated equivalently

involving in the same way N, for which by § 2 the conclusion follows. Also by putting $f(x) = 0$ in (13), we see that the hypothesis of the second part of (II) may be put in terms of N instead of K.

This leaves only (VI) to be proved. By (VI) of § 2 stated for N, with $\zeta = -1$, $m = 2$, we have

$$N_2(x, y) = \sum \frac{\varphi_p(x)\bar{\varphi}_p(y)}{(\lambda_p - \lambda_0)^2}, \tag{14}$$

the series converging uniformly in x, y. From (12),

$$K_2(x, y) = N_2(x, y) - 2\lambda_0 \int K(x, s) N_2(s, y)\, ds + \lambda_0^2 \int K_2(x, s) N_2(s, y)\, ds,$$

by means of which (14) yields

$$K_2(x, y) = \sum \frac{\varphi_p(x)\bar{\varphi}_p(y)}{\lambda_p^2}, \tag{15}$$

the series converging uniformly. Direct computation from (15) of the higher kernels gives the corresponding series for greater values of m. For $m = 1$, suppose the series converges uniformly, and write

$$A(x, y) = \frac{\varphi_p(x)\bar{\varphi}_p(y)}{\lambda_p}.$$

By use of the relation

$$\varphi_p(x) + \lambda_0 \int N(x, s)\varphi_p(s)\, ds = \frac{\lambda_p}{\lambda_p - \lambda_0} \varphi_p(x),$$

which is an immediate consequence of (12), we find

$$A(x, y) + \lambda_0 \int N(x, s) A(s, y)\, ds = \sum \frac{\varphi_p(x)\bar{\varphi}_p(y)}{\lambda_p - \lambda_0}.$$

This series also converges uniformly and hence represents $N(x, y)$. Thus

$$A(x, y) + \lambda_0 \int N(x, s) A(s, y)\, ds = N(x, y)$$

and $A(x, y) = K(x, y)$.

All the expected results have now been shown to hold in the present case; I defer their formal statement until a further extension is made.

4. *Straight Line not through Origin.* More generally, let

$$\bar{\alpha}K(x,y)+\alpha\bar{K}(y,x) = 2\int K(x,s)\bar{K}(y,s)ds$$

or else

$$\bar{\alpha}K(x,y)+\alpha\bar{K}(y,x) = 2\int \bar{K}(s,x)K(s,y)ds,$$

where $\alpha \neq 0$. Writing $K(x,y) = \alpha L(x,y)$, we see that L satisfies the condition stated for K in (4) or (5). Hence a cpv μ for L satisfies $R(\mu) = 1$, and a cpv λ for K satisfies $R(\alpha\lambda) = 1$.

The verification of all the other results of § 3 is easy. We have the following theorem.

If $K(x,y)$ is continuous and not identically zero, and satisfies one of the conditions

$$\bar{\alpha}K(x,y)+\alpha\bar{K}(y,x) = 2\int K(x,s)\bar{K}(y,s)ds,$$

$$\bar{\alpha}K(x,y)+\alpha\bar{K}(y,x) = 2\int \bar{K}(s,x)K(s,y)ds,$$

where $\alpha \neq 0$, it satisfies the other also; (I) *every cpv λ is on the straight line $R(\alpha\lambda) = 1$*; (II–VI) *the other properties stated in* § 1 *hold without change.*

5. *Conclusion.* We have given conditions on the kernel which will cause the cpv's to lie on any chosen line through the origin in § 2, and on any line not through the origin in § 4. The results of § 2 are not contained in those of § 4; but if we put $\alpha = ri/\sqrt{\zeta}$, where $|\zeta| = 1$, r is real and positive, we may view them as limiting cases for $r\to\infty$. A slight change of notation would of course make possible the inclusion of all the results in a single theorem.

CORNELL UNIVERSITY

ON THE POLYNOMIAL OF THE BEST APPROXIMATION TO A GIVEN CONTINUOUS FUNCTION*

BY J. SHOHAT (JACQES CHOKHATE)

1. *A Theorem on Minimizing Polynomials.* Let $f(x)$ and $p(x)$ be defined on a finite interval (a, b); $f(x)$ is bounded and integrable, $p(x)$ is integrable and not negative.

THEOREM I.† *If there exist two numbers α, β such that $a \leqq \alpha < \beta \leqq b$, and such that*

$$\int_c^d p(x)\,dx > 0$$

whenever $\alpha \leqq c < d \leqq \beta$, then there exists one and only one polynomial of degree $\leqq n$ minimizing the integral

$$I_{nk} = \int_a^b p(x)\,|f(x) - U_{nk}(x)|^k\,dx,$$

where

$$U_{nk}(x) = \sum_{i=0}^{n} u_{ik} x^i,$$

provided that $k > 1$. If $k = 1$, the proof of existence applies without change; and the approximating polynomial is unique, if $f(x)$ is continuous on (a, b), and if

$$\int_c^d p(x)\,dx > 0$$

whenever $a \leqq c < d \leqq b$.

The proof may be organized as follows.

* Presented to the Society, December 26, 1924. The author wishes to acknowledge with appreciation many helpful suggestions made by Professor D. Jackson in connection with this paper.

† Cf. G. Pólya, *Sur un algorithme ...*, COMPTES RENDUS, vol. 157 (1913), pp. 840–843; D. Jackson, *On functions of closest approximation*, TRANSACTIONS OF THIS SOCIETY, vol. 22 (1921), pp. 117–128, *Note on a class of polynomials of approximation*, ibid., vol. 22 (1921), pp. 320–326, *A generalized problem in weighted approximation*, ibid., vol. 26 (1924), pp. 133–154, *Note on the convergence of weighted trigonometric series*, this BULLETIN, vol. 29 (1923), pp. 259–263.

2. *Existence of a Minimizing Polynomial.* The value of n being regarded as fixed, take

$$\delta = \frac{\beta - \alpha}{8n^2}. \tag{1}$$

For $y \pm \delta$ belonging to (α, β), we have

$$\int_y^{y+\delta} p(x)\,dx, \quad \int_{y-\delta}^{y} p(x)\,dx \geqq h(\delta) = h_n, \tag{2}$$

since these integrals are continuous and positive functions of y; h_n depends on n only. For an arbitrarily large $A > 0$ we can find, according to Kirchberger,* certain K_i such that *any one* of the inequalities $|u_{ik}| > K_i$ $(i = 0, 1, 2, \cdots, n)$ implies

$$\max |U_{nk}(x)| \text{ in } (\alpha, \beta) = |U_{nk}(\xi)| = M > 2\Phi + 2\sqrt[k]{A/h_n}, \tag{3}$$
$$\Phi = \max |f(x)| \text{ in } (a, b), \qquad \alpha \leqq \xi \leqq \beta.$$

Using Markoff's theorem† and (1), we find, for $|x - \xi| \leqq \delta$,

$$|U_{nk}(x) - U_{nk}(\xi)| \leqq |x - \xi| \cdot \frac{2n^2 M}{\beta - \alpha} \leqq \frac{M}{4},$$

$$|U_n(x)| > \frac{M}{2} > \Phi + \sqrt[k]{A/h_n},$$

$$|f(x) - U_{nk}(x)| > \sqrt[k]{A/h_n}, \tag{4}$$

whence it follows that $I_{nk} > A$, since at least one of the intervals $(\xi - \delta, \xi)$, $(\xi, \xi + \delta)$ belongs to (α, β). Therefore we must take $|u_{ik}| \leqq K_i$ $(i = 0, 1, 2, \cdots, n)$, which proves the existence of a minimizing polynomial. We use the notation

$$M_{nk} = \min I_{nk} = \int_a^b p(x)\,|f(x) - P_{nk}(x)|^k\,dx. \tag{5}$$

---

* *Ueber Tchebychefsche Annäherungsmethoden*, Dissertation, Göttingen, 1902.

† Cf., e. g., M. Riesz, *Eine trigonometrische Interpolationsformel, ...*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 354-368; pp. 359-360.

3. *Uniqueness of the Solution.* Case I: $k>1$. It can be easily shown that

$$|\tfrac{1}{2}(x+y)|^k<\tfrac{1}{2}(|x|^k+|y|^k) \tag{6}$$

if $k>1$ and $x\neq y$. Assuming the existence of two non-identical solutions $\varphi_1(x)=f(x)-P'_{nk}(x)$, $\varphi_2(x)=f(x)-P''_{nk}(x)$, we get, using in (6) the function $\varphi_3(x)=\tfrac{1}{2}[\varphi_1(x)+\varphi_2(x)]$,

$$\int_a^b p(x)|\varphi_3(x)|^k dx<\tfrac{1}{2}M_{nk}+\tfrac{1}{2}M_{nk}=M_{nk}, \tag{7}$$

which is impossible.

Case II: $k=1$. Proceeding as above, we get the impossible inequality (7), unless $\varphi_1(x)\varphi_2(x)\geqq 0$ for $a\leqq x\leqq b$. Evidently every function of the type

$$\varphi_3(x)=h\varphi_1(x)+l\varphi_2(x)=\varphi_1(x)+l\psi(x), \tag{8}$$

where $\psi(x)=\varphi_2(x)-\varphi_1(x)$, $h,l>0, h+l=1$, is also a solution. Inasmuch as $\varphi_1(x)$ and $\varphi_2(x)$ can never have opposite signs, all roots of $\varphi_3(x)$ are the roots common to $\varphi_1(x)$ and $\varphi_2(x)$, or, what is the same, to $\varphi_1(x)$ and $\psi(x)$. Since $\psi(x)$ is a polynomial of degree $\leqq n$, we conclude:

*If there exist two solutions $\varphi_1(x)$, $\varphi_2(x)$, there exists necessarily a third one $\varphi_3(x)$, which has not more than n roots in (a,b). We shall prove that the last conclusion leads to a contradiction.**

Let the zeros of $\varphi_3(x)$ in (a,b) be

$$x_1<x_2<\cdots<x_m \qquad (m\leqq n). \tag{9}$$

We have necessarily $m>0$; otherwise the function $\varphi(x)=\varphi_3(x)+\eta$ with a properly chosen η gives $I_{nk}<M_{nk}$. Consider now two groups of the roots (9):

$$z_1, z_2, \cdots, z_{m'}; \quad y_1, y_2, \cdots, y_{m''} \qquad (m'+m''=m),$$

where the z's are the roots, if any, at which $\varphi_3(x)$ changes sign, and the y's are those at which it does not. Form the polynomial

* In the case that $f(x)\equiv x^{n+1}$, it was established in my Thesis that the number of roots is $n+1$.

$$\prod(x) = \pm\prod_{i=1}^{m}(x-z_i) \quad (\prod(x)=1 \text{ for } m'=0), \tag{10}$$

where the sign $\pm$ is chosen so that $\prod(x)\varphi_3(x)\geqq 0$ in (a, b). Draw the curve $y=|\prod(x)|$ and the line $y=\varepsilon$ with $\varepsilon>0$ sufficiently small. Project the points of intersection on the x-axis. Thus we get a set of points Z, made up of m' intervals

$$(z_i-\delta_i', z_i+\delta_i'') \quad (i=1, 2, \cdots, m'), \tag{11}$$

and a complementary set $K=b-a-Z$, such that

$$\varepsilon = \max\left|\prod(x)\right| \text{ on } Z = \min\left|\prod(x)\right| \text{ on } K. \tag{12}$$

It is evident that for ε sufficiently small all maxima points of the curve and all y's are outside Z, and

$$\int_K p(x)\,dx > 3\int_Z p(x)\,dx. \tag{13}$$

The numbers ε and δ_i', δ_i'' being fixed, we take $\delta>0$ so small that the set of points

$$Y: \quad y_j-\delta \leqq x \leqq y_j+\delta \quad (j=1, 2, \cdots, m'') \tag{14}$$

is outside Z, and

$$\begin{aligned} &\int_Y p(x)\,dx < \frac{\varepsilon}{2M}\left(\int_K p(x)\,dx - \int_Z p(x)\,dx\right), \\ &\int_Y p(x)\,dx < \int_Z p(x)\,dx, \end{aligned} \tag{15}$$

where M is the maximum of $|\prod(x)|$ in (a, b). Since $\varphi_3(x)$ is continuous and has no zeros in

$$K' = b-a-Z-Y = K-Y, \tag{16}$$

we have

$$|\varphi_3(x)| \geqq h' > 0 \text{ on } K'. \tag{17}$$

We form now the function

$$\varphi(x) = \varphi_3(x) - \eta\prod(x), \quad 0<\eta<h'/M, \tag{18}$$

and observe that

$$(19)\qquad \begin{cases} |\varphi(x)| = |\varphi_3(x)| - \eta \left|\prod(x)\right| & \text{on } K', \\ |\varphi(x)| \leqq |\varphi_3(x)| + \eta \left|\prod(x)\right| & \text{on } Z+Y, \end{cases}$$

whence, by combination with (12), (13), and (15), we get the desired contradiction:

$$\int_a^b p(x)|\varphi(x)|\,dx < \int_a^b p(x)|\varphi_3(x)|\,dx = M_{n1}.$$

The existence proof gives incidentally the following corollary.

COROLLARY. *To an arbitrarily large $A>0$ there correspond certain K_i $(i = 0, 1, 2, \cdots, n)$ such that any one of the inequalities $|u_i| > K_i$ implies that*

$$\int_a^b p(x)\left|f(x) - \sum_{i=0}^{n} u_i x^i\right|^k dx > A,$$

where $P(x)$ and $f(x)$ have the properties specified, and $k \geqq 1$.

4. *Tchebychef Approximating Polynomials.* THEOREM II. *Let $f(x)$ be continuous in (a, b), $p(x) \geqq 0$, and*

$$\int_\alpha^\beta p(x)dx > 0$$

whenever $a \leqq \alpha < \beta \leqq b$. Then, for $k \to \infty$, the minimizing polynomial $P_{nk}(x)$ tends uniformly to the polynomial $T_n(f)$, of degree $\leqq n$, which gives the closest approximation to $f(x)$ in the interval (a, b), in the sense of Tchebychef.

The proof is very similar to that given by G. Pólya, loc. cit. We assume, without loss of generality,

$$(20)\qquad \int_a^b p(x)dx = 1,$$

replacing, if necessary, $p(x)$ by $cp(x)$, $c>0$ being properly chosen. We have evidently, taking $k>1$,

$$(21)\qquad \begin{aligned} M_{nk} &= \min I_{nk} < E_n^k, \\ E_n &= \max |f(x) - T_n(f)| \text{ for } a \leqq x \leqq b. \end{aligned}$$

If $a < \alpha < \beta < b$, and $|f(x) - P_{nk}(x)|$ attains its minimum in (α, β) at $x = z$, then we get, using (21),

$$|P_{nk}(z)| < \frac{E_n}{\sqrt[k]{\int_a^b p(x)\,dx}} + \Phi < \frac{E_n}{\int_a^b p(x)\,dx} + \Phi,$$

(22)

$$\Phi = \max |f(x)| \text{ in } (a, b).$$

We form now $n+1$ partial intervals (α_i, β_i) all belonging to (a, b) and separated by segments of a certain length δ, and we use (22). *Since the right-hand member in* (22) *does not depend on* k, we can follow the reasoning of G. Pólya and prove (a) the coefficients in $P_{nk}(x)$, considered as functions of k, are bounded, so that the sequence (P_{nk}) admits one or several limiting polynomials; (b) all these limiting polynomials coincide with the polynomial of best approximation $T_n(f)$, which is known to be unique.

COROLLARY. *The polynomial minimizing the integral*

$$\int_a^b p(x)\,|x^n + \cdots|^k dx$$

tends for $k \to \infty$ *uniformly to*

$$\frac{1}{B}\cos(n \arccos y), \qquad y = \frac{2x-a-b}{b-a},$$

for every $p(x)$ *satisfying the conditions of Theorem* II, *where* $B(=2^{2n-1}/(b-a)^n)$ *is the coefficient of* x^n *in* $\cos(n \arccos y)$.

In fact, $x^n - (1/B)\cos(n \arccos y)$ is the polynomial of degree $\leq n-1$ giving the best approximation to x^n in (a, b).

The results given above hold if we replace polynomials by trigonometric sums, provided $p(x)$ and $f(x)$ are periodic functions.

THE UNIVERSITY OF CHICAGO

RESOLVENT SEXTICS OF QUINTIC EQUATIONS

BY L. E. DICKSON

1. *Introduction.* The object of this paper is to give simple derivations of the classic resolvents which have been obtained heretofore by elaborate computations.

Jacobi* established the form of a remarkable resolvent, but neither found the values of the coefficients nor gave the simple details (§ 2 below) which lead directly to that form.

Cayley† was not aware of Jacobi's work when he fully computed the same resolvent. Noting that its roots are functions of the differences of the roots x_i of the quintic, he first computed at length the resolvent sextic under the restriction that $x_5 = 0$. Then the coefficients were "completed by the introduction of the terms involving the constant coefficient of the quintic." No details were given of the latter long computation, which may perhaps be best made by utilizing the fact that the coefficients are seminvariants. The simple new method employed here (§ 3) makes initial use of the latter fact as well as of a lemma which reduces the search for the needed seminvariants of the quintic to a mere inspection of the invariants of a quartic.

From the Jacobi-Cayley resolvent (which is a simple transform of the old Malfatti resolvent) it is an immediate step (§ 5) to the noteworthy covariant resolvent discovered by Perrin,‡ and independently by McClintock,§ each time as the final step of a long computation.

---

* JOURNAL FÜR MATHEMATIK, vol. 13 (1835), pp. 340–52; WERKE, vol. 3, 1884, pp. 269–84.

† PHILOSOPHICAL TRANSACTIONS, London, vol. 151 (1861), pp. 263–76; COLLECTED MATHEMATICAL PAPERS, vol. 4, pp. 309–24.

‡ COMPTES RENDUS DU DEUXIÈME CONGRÈS INTERNATIONAL DES MATHÉMATICIENS, Paris, 1902, pp. 199–223. Announced in BULLETIN DE LA SOCIÉTÉ DE FRANCE, vol. 11 (1882–83), pp. 64–65.

§ AMERICAN JOURNAL, vol. 8 (1886), pp. 45–84; vol. 20 (1898), pp. 157–192.

2. *The Symmetric Functions of* $z_1, \cdots, z_6$. Writing ij for x_ix_j, we consider the function

$$12345 = 12+23+34+45+51.$$

It is unaltered by the substitutions $a=(12345)$, $b=(25)(34)$. Since $b^{-1}ab = a^{-1}$, a and b generate a group of ten even substitutions. Since 12345 is therefore unaltered by these ten, it takes at most 60/10 distinct values under the group G of all 60 even substitutions. It actually takes the six distinct values given by the first (positive) parts of

$$z_1 = 12345-13524, \quad z_2 = 12453-14325,$$
$$z_3 = 12534-15423, \quad z_4 = 15243-12354,$$
$$z_5 = 14235-12543, \quad z_6 = 13254-12435.$$

In fact, (345), (354), (253), (243), (23)(45) replace z_1 by $z_2, \cdots, z_6$. Hence $z_1, \cdots, z_6$ are merely permuted by each of the 60 even substitutions. Next, every odd substitution O replaces each z_i by the negative of some z. For, (2354) replaces z_1 by $-z_1$. Let E be one of the even substitutions which replaces z_i by z_1 and write E_1 for the even substitution $(2354)^{-1}E^{-1}$. Then $O = E(2354)E_1$ replaces z_i by the function by which E_1 replaces $-z_1$ and that function is the negative of some z.

Hence any homogeneous symmetric function of $z_1, \cdots, z_6$ of even degree is symmetric in $x_1, \cdots, x_5$. But if it is of odd degree in the z's, it merely changes sign when any two x's are interchanged and hence is divisible by the product of the ten differences of the x's, the quotient being symmetric in the x's.

3. *The Jacobi-Cayley Resolvent.* The discriminant Δ of

$$f(x,y) = a_0x^5+5a_1x^4y+10a_2x^3y^2+10a_3x^2y^3+5a_4xy^4+a_5y^5$$

is defined to be the polynomial such that $5^5a_0^{-8}\Delta$ is equal to the product of the squares of the ten differences of the roots x_i of $f(x,1)=0$.

In the sextic having the roots $z_1, \cdots, z_6$, the coefficients of z^5 and z^3 are zero by § 2, being of odd degrees 1 and 3

in the z's, so that their degrees in the x's are less than the degree 10 of the product H of the differences of the x's. The coefficient of z is the product of a numerical constant by H or by $a_0^{-4}\sqrt{\Delta}$. It is convenient to multiply the sextic by a_0^6. We get

$$a_0^6 z^6 + a_0^4 u_1 z^4 + a_0^2 u_2 z^2 + u_3 = \nu a_0^2 \sqrt{\Delta}\, z,$$

where ν is a numerical constant, while $a_0^{-2i} u_i$ is the sum of the products of the z's taken $2i$ at a time and hence is of total degree $4i$ in the x's and of degree $2i$ in any one root x. By § 2, it is symmetric in the x's. It is expressible as a polynomial in the differences of the x's, since

$$z_1 = (1-5)(2-5) + (2-5)(3-5) + (3-5)(4-5) - (2-5)(4-5) - (4-5)(1-5) - (1-5)(3-5).$$

It follows* that u_i is a seminvariant of f of degree $2i$ and weight $4i$. By a seminvariant S of f is meant a homogeneous isobaric polynomial in $a_0, \cdots, a_5$ for which $\Omega S \equiv 0$, i. e., is annihilated by the operator

$$\Omega = a_0 \frac{\partial}{\partial a_1} + 2a_1 \frac{\partial}{\partial a_2} + 3a_2 \frac{\partial}{\partial a_3} + 4a_3 \frac{\partial}{\partial a_4} + 5a_4 \frac{\partial}{\partial a_5}.$$

Since u_1 is of degree 2 and weight 4, it lacks a_5 and is the product of I by a numerical constant, as shown by the following lemma.

LEMMA. *If a seminvariant S of the quintic $f(x, y)$ lacks a_5, it is a seminvariant of the quartic*

$$q = a_0 x^4 + 4a_1 x^3 y + 6a_2 x^2 y^2 + 4a_3 x y^3 + a_4 y^4.$$

If the weight of S is double its degree, it is an invariant of q and hence is a polynomial in

$$I = a_0 a_4 - 4a_1 a_3 + 3a_2^2,$$

$$J = a_0 a_2 a_4 - a_0 a_3^2 + 2a_1 a_2 a_3 - a_1^2 a_4 - a_2^3.$$

For, S is homogeneous and isobaric and is annihilated by the operator derived from Ω by suppressing the final

* Dickson, *Algebraic Invariants*, New York, 1914, p. 53.

derivative. Hence S is a seminvariant of q. A seminvariant of degree d and weight w of a binary form of order p is the leader of a unique covariant of order $pd-2w$ (*Invariants*, p. 43 and Ex. 1, p. 40). It is therefore an invariant if $p=4$, $w=2d$.

A known seminvariant of the same degree 4 and weight 8 as u_2 is

$$T = a_0^2a_3a_5-3a_0a_1a_2a_5-5a_0a_1a_3a_4+10a_0a_2^2a_4-4a_0a_2a_3^2 \\ +2a_1^2a_5-5a_1^2a_2a_4+14a_1^2a_3^2-16a_1a_2^2a_3+6a_4^2$$

(it suffices to verify that $\Omega T\equiv 0$). We delete the term $a_0^2a_3a_5$ from u_2 by subtracting a multiple of T. In the resulting seminvariant v, the only terms involving a_5 are those in $a_5(\alpha a_0a_1a_2+\beta a_1^3)$, since the terms in parenthesis, together with the deleted term $a_0^2a_3$, are the only possible terms of degree 3 and weight 3. By inspection, Ωv is the sum of

$$\alpha a_0^2a_2a_5+(2\alpha+3\beta)a_0a_1^2a_5$$

and terms free of a_5. Hence $\alpha=\beta=0$. Thus v lacks a_5 and by the Lemma is an invariant of q and hence is a product of I^2 by a constant. Thus u_2 is a linear combination of I^2 and T.

To determine u_3 we shall employ the seminvariant P of the same degree 6 and same weight 12 (cf. § 6):

$$P = a_0^2a_2a_5^2-2a_0^2a_3a_4a_5+a_0^2a_4^3-a_0a_1^2a_5^2-4a_0a_1a_2a_4a_5 \\ +8a_0a_1a_3^2a_5-2a_0a_1a_3a_4^2-2a_0a_2^2a_3a_5+14a_0a_2^2a_4^2 \\ -22a_0a_2a_3^2a_4+9a_0a_3^4+6a_1^2a_4a_5-12a_1^2a_2a_3a_5-15a_1^2a_2a_4^2 \\ +10a_1^2a_3^2a_4+6a_1a_2^2a_5+30a_1a_2^2a_3a_4-20a_1a_2a_3^3 \\ -15a_2^4a_4+10a_2^3a_3^2.$$

We use a_0P and TI to delete the terms $a_0^3a_2a_5^2$ and $a_0^3a_3a_4a_5$ from u_3. In the resulting seminvariant S, the terms having the factor a_5 are those in

$$\alpha a_0^2a_1^2a_5^2+\beta a_0^2a_1a_2a_4a_5+\gamma a_0^2a_1a_3^2a_5+a a_0^2a_2^2a_3a_5 \\ +ba_0a_1^3a_4a_5+ca_0a_1^2a_2a_3a_5+da_0a_1a_2^3a_5+ea_1^4a_3a_5+ga_1^3a_2^2a_5.$$

These terms alone furnish the part of ΩS involving a_5:

$$2\alpha a_0^3 a_1 a_5^2 + \beta a_0^3 a_2 a_4 a_5 + \gamma a_0^3 a_3^2 a_5 + (3b + 2\beta + 10\alpha) a_0^2 a_1^2 a_4 a_5 + (2c + 4a + 4\beta + 6\gamma) a_0^2 a_1 a_2 a_3 a_5 + (d + 3a) a_0^2 a_2^2 a_5 + (2c + 4b + 4e) a_0 a_1^3 a_3 a_5 + (3c + 3g + 6d) a_0 a_1^2 a_2^2 a_5 + (4g + 3e) a_1^4 a_2 a_5.$$

The conditions that this be zero identically require that $\alpha, \beta, \gamma, a, b, c, d, e, g$ shall all vanish. Since S therefore lacks a_5, the lemma shows that it is a sum of terms $I^r J^s$ whose degree is $2r + 3s = 6$, whence either $s = 0$, $r = 3$ or $s = 2$, $r = 0$. Thus u_3 is a linear combination of $I^3, J^2, a_0 P, TI$. Hence the sextic is of the form

$$a_0^6 z^6 + a_0^4 \alpha I z^4 + a_0^2 (\beta I^2 + \gamma T) z^2 + \delta I^3 + \epsilon J^2 + \lambda a_0 P + \mu TI = a_0^2 \nu \sqrt{\Delta} z,$$

where the Greek letters are numerical constants. To find their values, we employ the special quintic having $x_3 = -x_1$, $x_4 = -x_2$, $x_5 = 0$:

$$x(x^2 - x_1^2)(x^2 - x_2^2) = x^5 + 10 a_2 x^3 + 5 a_4 x = 0,$$

where $10 a_2 = -x_1^2 - x_2^2$, $5 a_4 = x_1^2 x_2^2$. Then

$$z_1 = z_3 = (x_1 + x_2)^2, \qquad z_2 = z_4 = -(x_1 - x_2)^2,$$
$$z_5 = x_1^2 - x_2^2 - 4 x_1 x_2, \qquad z_6 = x_2^2 - x_1^2 - 4 x_1 x_2.$$

Using temporarily the abbreviations

$$s = x_1^2 + x_2^2, \qquad p = x_1^2 x_2^2, \qquad t = x_1 x_2,$$

we see that z_1 and z_2 are the roots of $z^2 - 4tz + 4p - s^2 = 0$, while z_5 and z_6 are the roots of $z^2 + 8tz + 20p - s^2 = 0$. Hence the sextic is

$$z^6 - (20p + 3s^2) z^4 + (240p^2 - 8ps^2 + 3s^4) z^2 + (4p - s^2)^2 (20p - s^2) = 128 tp (4p - s^2) z.$$

Replacing s, p, t by their values in terms of a_2, a_4, we get

$$z^6 - 100(a_4 + 3a_2^2) z^4 + 2000(3a_4^2 - 2a_4 a_2^2 + 15 a_2^4) + 40000(a_4^3 - 11 a_4^2 a_2^2 + 35 a_4 a_2^4 - 25 a_2^6) = 12800 a_4 (a_4 - 5a_2^2) \sqrt{5 a_4} z.$$

But for $a_0 = 1$, $a_1 = a_3 = a_5 = 0$, we have

$$I = a_4 + 3a_2^2, \qquad J = a_2a_4 - a_2^3, \qquad T = 10a_2^2a_4 + 6a_2^4,$$
$$\Delta = 16^2a_4^3(a_4 - 5a_2^2)^2, \qquad P = a_4^3 + 14a_2^2a_4^2 - 15a_2^4a_4.$$

We see at once that $\alpha = -100$, $\beta = 6000$, $\gamma = -4000$, $\nu = 800\sqrt{5}$. By the terms free of z,

$$\begin{aligned} \delta + \lambda &= 40000, & 9\delta + \varepsilon + 14\lambda + 10\mu &= -11 \cdot 40000, \\ & & 27\delta - 2\varepsilon - 15\lambda + 36\mu &= 35 \cdot 40000, \\ & & 27\delta + \varepsilon + 18\mu &= -25 \cdot 40000. \end{aligned}$$

Hence $\lambda = 40000$, $\varepsilon = -25\lambda$, $\mu = \delta = 0$. The resolvent sextic is therefore

$$a_0^6z^6 - 100a_0^4Iz^4 + 2000a_0^2(3I^2 - 2T)z^2 - 800\sqrt{5}a_0^2\sqrt{\Delta}z + 40000(a_0P - 25J^2) = 0.$$

4. *Canonizant C.* The covariant of f,

$$C = Jx^3 + J_1x^2y + J_2xy^2 + J_3y^3,$$

having the leader J, may readily be found by means of the annihilator (*Invariants*, p. 39)

$$O = 5a_1\frac{\partial}{\partial a_0} + 4a_2\frac{\partial}{\partial a_1} + 3a_3\frac{\partial}{\partial a_2} + 2a_4\frac{\partial}{\partial a_3} + a_5\frac{\partial}{\partial a_4}.$$

We get

$$OJ = J_1 = \begin{vmatrix} a_0 & a_1 & a_3 \\ a_1 & a_2 & a_4 \\ a_2 & a_3 & a_5 \end{vmatrix}, \qquad \tfrac{1}{2}OJ_1 = J_2 = \begin{vmatrix} a_0 & a_2 & a_3 \\ a_1 & a_3 & a_4 \\ a_2 & a_4 & a_5 \end{vmatrix},$$

$$\tfrac{1}{3}OJ_2 = J_3 = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}, \qquad J = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}.$$

Hence

$$C = \begin{vmatrix} a_0x + a_1y & a_1x + a_2y & a_2x + a_3y \\ a_1x + a_2y & a_2x + a_3y & a_3x + a_4y \\ a_2x + a_3y & a_3x + a_4y & a_4x + a_5y \end{vmatrix}.$$

The name canonizant is given to C since its three linear

factors u, v, w have the property* that $f = u^5 + v^5 + w^5$. For a direct derivation of C, see § 6. An elementary verification that C is a covariant may be made by manipulating a single determinant. Under the transformation $x = X + \varepsilon Y$, $y = Y$, f becomes $F = A_0X^5 + 5A_1X^4Y + \cdots$, where†

$$\begin{aligned}
A_0 &= a_0, \\
A_1 &= a_1 + \varepsilon a_0, \\
A_2 &= a_2 + 2a_1\varepsilon + a_0\varepsilon^2, \\
A_3 &= a_3 + 3a_2\varepsilon + 3a_1\varepsilon^2 + a_0\varepsilon^3, \\
A_4 &= a_4 + 4a_3\varepsilon + 6a_2\varepsilon^2 + 4a_1\varepsilon^3 + a_0\varepsilon^4, \\
A_5 &= a_5 + 5a_4\varepsilon + 10a_3\varepsilon^2 + 10a_2\varepsilon^3 + 5a_1\varepsilon^4 + a_0\varepsilon^5.
\end{aligned}$$

To the elements of the second row of C add the products of those of the first row by ε. To the elements of the third row add the products of those of the first row by ε^2 and those of the original second row by 2ε. Replace x and y by their values. To the new determinant apply the corresponding operations on columns instead of rows. We get a determinant of type C written in capital letters. Finally, the interchange of x with y and hence of a_0 with a_5, a_1 with a_4, a_2 with a_3, replaces C by a determinant which reduces to C by writing its rows in reverse order and then the columns in reverse order.

5. *Covariant Resolvent.* We employ the linear covariant $L = Px + Qy$, where P was defined in § 3 and Q is derived from P by the substitution $(a_0a_5)(a_1a_4)(a_2a_3)$ induced by the interchange of x and y. The constant term $40000(a_0P - 25J^2)$ of the resolvent in § 3 is therefore the leader of the covariant $K = 40000(fL - 25C^2)$ of order 6. Equated to zero, it gives the covariant resolvent of Perrin, which was rediscovered by McClintock and called the central resolvent.

* Salmon, *Modern Higher Algebra*, 4th ed., p. 153; German translation by Fiedler, p. 199.

† We may write symbolically a^i for a_i, $f = (x + a_1y)^5$. Then $F = (X + \beta Y)^5$, $\beta = a_1 + \varepsilon$. After expansion, the terms free of a_1 are to be multiplied by a_0. From β^i we get A_i.

For its use as a resolvent of the quintic $f(x, 1) = 0$, it is essential to know the expressions for its roots in terms of $x_1, \cdots, x_5$. To find these expressions, we shall give a process indicated without proof by McClintock, but requiring correction by inserting factors a_0. Write

$$\Phi_1 = a_0 z_1 = a_0 \sum x_1 (x_2 - x_3),$$

where (and below) each of the five terms of $\sum$ are derived from the preceding term by the substitution (12345). Replacing each x_i by its reciprocal, we get

$$a_0 \sum \frac{(x_3 - x_2) x_4 x_5}{x_1 x_2 x_3 x_4 x_5} = -\frac{a_0}{a_5} \Psi_1, \quad \Psi_1 = a_0 \sum (x_3 - x_2) x_4 x_5.$$

Similarly, from Ψ_1 we get $-\Phi_1 a_0 / a_5$. We shall prove that

$$G = \prod_{i=1}^{6} (\Phi_i x - \Psi_i y)$$

is a covariant of f, where $\Phi_i = a_0 z_i$, and Ψ_i is derived from Φ_i by replacing each root x by its reciprocal. Since the leader of G is $\Phi_1 \cdots \Phi_6$, which is equal to the constant term of the resolvent in § 3, and hence to the leader of K, it will follow that $G = K$.

Apply transformation $x = X, y = X$ to $f = a_0 \prod (x - x_1 y)$. We get $F = a_5 \prod (X - x_1^{-1} Y)$. The function Φ_1 for F is

$$a_5 \sum x_1^{-1} (x_2^{-1} - x_3^{-1}) = \frac{a_5}{a_0} \left(-\frac{a_0}{a_5} \Psi_1 \right) = -\Psi_1.$$

The function Ψ_1 for F is the product of a_5/a_0 by the function $-\Phi_1 a_0 / a_5$ obtained above from Ψ_1 by replacing each x_i by its reciprocal. Hence G for F is

$$\prod (-\Psi_1 X + \Phi_1 Y) = \prod (\Phi_1 x - \Psi_1 y) = G.$$

Next, apply transformation $x = X + tY, y = Y$ to f. We get $F = a_0 \prod (X - X_1 Y)$, where $X_1 = x_1 - t$. The function Φ_1 for F is the seminvariant Φ_1 itself. The function Ψ_1 for F is

$$a_0 \sum (x_3 - x_2)(x_4 - t)(x_5 - t) = \Psi_1 - t\Phi_1 \equiv M_1.$$

Hence G for F is

$$\prod(\Phi_1 X - M_1 Y) = \prod[\Phi_1 x - (t\Phi_1 + M_1)y] = G.$$

Thus *the covariant resolvent* $K(x, 1) = 0$ *has the roots* Ψ_i/Φ_i.

A like process enables us to write down at once the linear factors of a covariant of order n whose leader is a seminvariant which is the product of n rational functions of the x's.

6. *Another Derivation of C and L.* If in a covariant φ of f we replace $x^r y^s$ by $(-1)^s \partial^{r+s}/(\partial y^r \partial x^s)$, i. e. replace the products of powers of x and y by symbolic products of powers of $\partial/\partial y$ and $-\partial/\partial x$, and apply the resulting operator to another covariant ψ of f, we obtain a covariant $[\varphi, \psi]$ of f (*Invariants*, top p. 61).

The quintic f has the covariant*

$$i = Ix^2 + I_1 xy + I_2 y^2, \quad I_1 = OI = a_0 a_5 - 3a_1 a_4 + 2a_2 a_3.$$
$$I_2 = \tfrac{1}{2} OI_1 = a_1 a_5 - 4a_2 a_4 + 3a_3^2.$$

Then

$$-\tfrac{1}{60}[i, f] = C, \quad -\tfrac{1}{2}[i, C] = L = Px + Qy.$$

THE UNIVERSITY OF CHICAGO

* It is the invariant I of the fourth polar of f.

AN ACKNOWLEDGEMENT

BY J. H. M. WEDDERBURN

Professor G. Scorza has kindly called my attention to the fact that the result of my note entitled *A theorem on simple algebras* (this BULLETIN, vol. 31, pp. 11–13) was given by him in his book, *Corpi Numerici e Algebre* (1921), pp. 346–352. I regret that I was unaware of this at the time the paper was published, and I take this means of acknowledging Professor Scorza's priority.

IMPROPER DOUBLE INTEGRALS

BY C. A. SHOOK

1. *Introduction.* It can be shown by examples that there exist integrals of the type

$$\int_a^c dy \int_b^d f(x, y)\, dx$$

where a, b, c and d are constants, in which an alteration of value occurs when the order of integration is reversed. Jordan* has given the following example. Let

$$f(x, y) = \frac{\partial^2 V}{\partial x\, \partial y},$$

where $V = \arctan(y/x)$; and let $a = b = 0$, and $c = d = 1$. I adduce one more example in which

$$f(x, y) = 2\frac{x-y}{(x+y)^3},$$

and in which the limits are the same as in the previous example. In this example, the value obtained is -1 or $+1$ according as we integrate first with respect to x or with respect to y.

Conditions sufficient to change the order of integration have been investigated by de la Vallée Poussin,† who showed that the whole question rests on the *uniform convergence* of the improper integrals involved. The question has also been studied by Brunel,‡ Jordan,* Stolz,§ and Osgood.‖

---

* *Cours d'Analyse*, vol. 2, pp. 66 et seq.

† *Sur la convergence des intégrales définies*, JOURNAL DE MATHÉMATIQUES, (4), vol. 8 (1892), p. 421.

‡ ENCYKLOPÄDIE DER MATHEMATISCHEN WISSENSCHAFTEN, II, 1, A, 1, p. 135.

§ *Differential- und Integralrechnung*, vol. 3, Chapter 16.

‖ ANNALS OF MATHEMATICS, (2), vol. 3, p. 129.

The following theorem is due to Osgood:

Let $f(x, y)$ be continuous in the region $a \leqq x$, $b \leqq y$, and let

(1) $\int_b^\infty f(x, y)\, dy$ *converge uniformly in an arbitrary finite interval,* $a \leqq x \leqq G$;

(2) $\int_a^\infty f(x, y)\, dx$ *converge uniformly in an arbitrary finite interval,* $b \leqq y \leqq H$;

(3) $\int_a^\infty dx \int_b^y f(x, y)\, dy$ *converge uniformly in the infinite interval,* $b \leqq y \leqq \infty$.

Then $\int_a^\infty dx \int_b^\infty f(x, y)\, dy$ and $\int_b^\infty dy \int_a^\infty f(x, y)\, dx$ both converge and are equal.

The purpose of this paper is to show that if we introduce the hypothesis that $f(x, y)$ does not change sign in the region, then (3) above can be replaced by the condition that

(3′) $$\int_a^\infty dx \int_b^\infty f(x, y)\, dy \text{ converge.}$$

Next, the theorem as thus modified will be extended to include the case in which the integrand does not remain finite at either or both lower limits. Finally the extended theorem will be applied to an important iterated integral arising in the theory of the gamma function.

2. *Definitions and Lemma.* Let $f(x, y)$ be continuous, $c \leqq x \leqq d$, $a < y$. Suppose that $f(x, y)$ does not remain finite as y tends to a. Finally, let $b > a$. Then the integral, $\int_b^\infty f(x, y)\, dy$, (or the integral $\int_a^b f(x, y)\, dx$), is said to converge uniformly in the interval $c \leqq x \leqq d$, if, corresponding to an arbitrarily small positive quantity ϵ, there exists a positive quantity Y, (or a positive quantity δ) *independent of* x, such that for every value of x in its interval,

$$\left| \int_{y_1}^{y_2} f(x, y)\, dy \right| < \epsilon, \text{ provided } y_1, y_2 \geqq Y, \text{ (or } a \leqq y_1, y_2 \leqq a + \delta).$$

To establish uniform convergence we have recourse to the Vallée-Poussin* μ-test. This may be stated as follows.

* Loc. cit.

If there exists a function $\mu(y)$ *such that*

(A) $|f(x, y)| \leqq \mu(y)$ *for all values of* x *and* y *in the region considered*;

(B) $\int_b^\infty \mu(y)\,dy$ $\left(or \int_a^b \mu(y)\,dy\right)$ *converges*;

then $\int_b^\infty f(x, y)\,dy$ $\left(or \int_a^b f(x, y)\,dy\right)$ *converges uniformly,* $c \leqq x \leqq d$.

We can now state and prove the following lemma.

LEMMA. *If*

(a) $f(x, y)$ *is continuous,* $a \leqq x \leqq b$, $c \leqq y$;

(b) $\int_c^\infty f(x, y)\,dy$ *converges uniformly,* $a \leqq x \leqq b$;

then $\int_a^b dx \int_c^\infty f(x, y)\,dy = \int_c^\infty dy \int_a^b f(x, y)\,dx$.

Since the integral $\int_c^\infty f(x, y)\,dy$ converges uniformly, $a \leqq x \leqq b$, it represents a continuous function of x in that interval, so that the integral

$$\int_a^b dx \int_c^\infty f(x, y)\,dy$$

has a definite meaning. Let n be any number greater than c and write

$$\int_a^b dx \int_c^\infty f(x, y)\,dy - \int_a^b dx \int_c^n f(x, y)\,dy = \int_a^b dx \int_n^\infty f(x, y)\,dy.$$

Also, the absolute value of the left hand member is equal to the absolute value of the right hand member. But, by (*b*), we have

$$\left|\int_n^\infty f(x, y)\,dy\right| < \varepsilon,\ n > m, \quad m \text{ independent of } x.$$

where ε is an arbitrarily small positive number and m is a positive number corresponding to ε. Also

$$\int_a^b dx \int_c^n f(x, y)\,dy = \int_c^n dy \int_a^b f(x, y)\,dx,$$

since both are equal to a certain definite volume. Hence

$$\left| \int_a^b dx \int_c^\infty f(x,y)\,dy - \int_c^n dy \int_a^b f(x,y)\,dx \right| < \epsilon(b-a), \quad n > m,$$

and this proves the lemma.

3. *The Modified Theorem.* In proving the modified theorem, use is made of an important theorem of Osgood* which may be stated as follows:

Let $F(x,y)$ be a function of the two independent variables x and y, satisfying the following conditions:

(1) $\lim_{x\to a} F(x,y)$ *exists, say equals* $f(y)$;

(2) $\lim_{y\to b} F(x,y)$ *exists, say equals* $g(x)$;

(3) $F(x,y)$ *converges uniformly when* x *tends to* a, *that is,* $|F(x',y) - F(x'',y)| \leq \epsilon$, $a \leqq x', x'' \leqq a+\delta$, δ *independent of* y.

Then we conclude

(4) $\lim_{y\to b} f(y)$ *exists, say equals* A;

(5) $\lim_{x\to a} g(x)$ *exists, say equals* B;

(6) $A = B$.

The theorem, with the proper modification of the third condition, holds when a and b are infinite.

Suppose now that $f(x,y)$ remains always positive, and let

$$F(x,y) = \int_a^x dx \int_b^y f(x,y)\,dy = \int_b^y dy \int_a^x f(x,y)\,dx.$$

We will show that $F(x,y)$ satisfies the conditions of the theorem just stated, taking the a and b of the theorem infinite. We have

$$\lim_{x\to\infty} \int_a^x dx \int_b^y f(x,y)\,dy = \int_a^\infty dx \int_b^y f(x,y)\,dy$$
$$= \int_b^y dy \int_a^\infty f(x,y)\,dx, \text{ by the Lemma;}$$

$$\lim_{y\to\infty} \int_b^y dy \int_a^x f(x,y)\,dx = \int_b^\infty dy \int_a^x f(x,y)\,dx$$
$$= \int_a^x dx \int_b^\infty f(x,y)\,dy, \text{ by the Lemma.}$$

* *Funktionentheorie*, vol. 1, p. 519. This theorem is fundamental in the consideration of double limits, on which see Osgood, this BULLETIN, vol. 3, p. 59; also *Funktionentheorie*, vol. 1, p. 66.

We now have to show that $F(x,y)$ converges uniformly when x increases without limit, that is, that

$$\left|\int_a^{x'} dx \int_b^y f(x,y)\,dy - \int_a^{x''} dx \int_b^y f(x,y)\,dy\right| < \epsilon$$

provided x' and x'' are both greater than a certain suitably chosen number. Combining integrals, we have

$$\left|\int_{x'}^{x''} dx \int_b^y f(x,y)\,dy\right| < \epsilon.$$

This is precisely the condition that the integral

$$\int_a^\infty dx \int_b^y f(x,y)\,dy$$

should converge uniformly. To apply the μ-test, take

$$\mu(x) = \int_b^\infty f(x,y)\,dy,$$

and notice that the conditions of the test are satisfied. Hence

$$\lim_{y\to\infty} \int_b^y dy \int_a^\infty f(x,y)\,dx = \lim_{x\to\infty} \int_a^x dx \int_b^\infty f(x,y)\,dy$$

or

$$\int_b^\infty dy \int_a^\infty f(x,y)\,dx = \int_a^\infty dx \int_b^\infty f(x,y)\,dy$$

which proves the modified theorem.

4. *The Extended Theorem.* We may now state a more extended theorem.

THEOREM A. *Let $f(x,y)$ be a function of x and y satisfying the conditions*

(1) *$f(x,y)$ is continuous, $a < x$, $b < y$, and may increase without limit as x tends to a or as y tends to b;*

(2) *$f(x,y) \geqq 0$ in the region considered;*

(3) $\int_a^c f(x,y)\,dx$ *converges uniformly,* $b < t \leqq y$, $\quad c > a$;

(4) $\int_c^\infty f(x,y)\,dx$ *converges uniformly,* $b < t \leqq y$, $\quad c > a$;

(5) $\int_b^d f(x,y)\,dy$ *converges uniformly,* $a < s \leqq x$, $\quad d > b$;

(6) $\int_d^\infty f(x,y)\,dy$ *converges uniformly,* $a < s \leqq x$, $\quad d > b$;

(7) $\int_a^\infty dx \int_b^\infty f(x, y)\, dy$ *converges.*

Then the order of integration can be reversed, that is,

$$\int_a^\infty dx \int_b^\infty f(x, y)\, dy = \int_b^\infty dy \int_a^\infty f(x, y)\, dx.$$

To prove this theorem, we write, omitting the integrand for brevity,

$$\int_a^\infty \int_b^\infty = \int_a^c \int_b^d + \int_a^c \int_d^\infty + \int_c^\infty \int_b^d + \int_c^\infty \int_d^\infty$$

The last of the four integrals of the right hand member falls directly under the modified theorem, stated in § 1 and proved in § 3. The other three integrals can be dealt with in a precisely similar way by making obvious changes. The hypotheses under Theorem A suffice for these proofs. But when the order of integration is changed in each of the four integrals, their sum becomes

$$\int_b^\infty \int_a^\infty$$

and this proves the theorem.

5. *Application of Theorem* A *to a Particular Integral.* In obtaining the relation between beta and gamma functions,*

$$B(m, n) = \frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)}$$

it is necessary to change the order of integration of the integral

$$\int_0^\infty dy \int_0^\infty x^{m-1} y^{m+n-1} e^{-y(1+x)}\, dx$$

where m and n are any positive numbers. This change of order can be justified by Theorem A. To apply the theorem take $c = d = 1$. Conditions 1, 2 and 7 are obviously satisfied. To show that conditions 3 to 6 inclusive

* Byerly, *Integral Calculus*, p. 113.

are satisfied we have to exhibit a set of μ-functions covering the various cases. Such a set follows:

	$0<t\leqq y\leqq 1$	$1\leqq y$
$\int_0^1 f(x,y)\,dx$	$\mu(x) = x^{m-1}e^{-tx}G,\ 1<m+n;$ $= x^{m-1}e^{-tx}H,\ 0<m+n<1$	$\mu(x) = x^{m-1}e^{-x}G$
$\int_1^\infty f(x,y)\,dx$	$\mu(x) = x^{m-1}e^{-tx}G,\ 1<m+n;$ $= x^{m-1}e^{-tx}H,\ 0<m+n<1$	$\mu(x) = x^{m-1}e^{-x}G$
	$0<s\leqq x\leqq 1$	$1\leqq x$
$\int_0^1 f(x,y)\,dy$	$\mu(x) = y^m e^{-y}M,\ 1<m;$ $= y^{m+n-1}e^{-y}N,\ 0<m\leqq 1$	$\mu(y) = y^m e^{-y}M,\ 1<m$ $= y^{m+n-1}e^{-y},\ 0<m\leqq 1$
$\int_1^\infty f(x,y)\,dy$	$\mu(x) = y^{m+n-1}e^{-y}M,\ 1<m;$ $= y^{m+n-1}e^{-y}N,\ 0<m\leqq 1$	$\mu(y) = y^{m+n-1}e^{-y}M$

$$G = \left(\frac{\lfloor m+n-1 \rfloor}{e}\right)^{\lfloor n-1 \rfloor m+}, \qquad H = e^{-t}t^{m+n-1};$$

$$M = \left(\frac{\lfloor m-1 \rfloor}{e}\right)^{\lfloor m-1 \rfloor}, \qquad N = s^{m-1};$$

$f(x, y)$ denotes the integrand of the integral under consideration.

YALE UNIVERSITY

THE FUNDAMENTAL REGION FOR A FUCHSIAN GROUP*

BY L. R. FORD

1. *Introduction.* The present paper is an attempt to lay the groundwork of the theory of Fuchsian groups by basing the treatment on concepts of a very simple sort. The fundamental region to which we are led is not new. It is given by the Fricke-Klein method† under certain circumstances and is identical with that given by Hutchinson‡ in an important paper. However, we make use neither of non-euclidean geometry nor of quadratic forms, and we are able to derive the major results of the theory of Fuchsian groups in an unexpectedly simple manner.

2. *The Group.* Given a group of linear transformations with an invariant circle or straight line K', the interior of K' (or the half-plane on one side of K') being transformed into itself by each transformation of the group. We shall assume that there exists a point A, not on K', such that there are no points congruent to A in a sufficiently small neighborhood of A.

Let G be a linear transformation carrying K' into the unit circle K with center at the origin and carrying A to the origin. Let S be any transformation of the group; then the set of transformations

$$T = GSG^{-1}$$

is a group with K as principal circle.§ Configurations

* Presented to the Society, September 11, 1925.

† Fricke-Klein, *Vorlesungen über die Theorie der automorphen Funktionen*, vol. I, Chap. II.

‡ J. I. Hutchinson, *A method for constructing the fundamental region of a discontinuous group of linear transformations*, TRANSACTIONS OF THIS SOCIETY, vol. 8 (1907), pp. 261–269.

§ We use this order to mean the transformation G^{-1}, followed by S, followed by G. That is, writing $z' = S(z)$, $z' = G(z)$, etc., as the combining transformations, the new transformation is $z' = T(z) = G\{S[G^{-1}(z)]\}$.

which are congruent by transformations of the new group are carried by G^{-1} into configurations which are congruent by transformations of the original group. It will suffice, then, to find a fundamental region for the new group.

The condition that

$$z' = T(z) = \frac{az+b}{cz+d}$$

leave the unit circle unchanged is that $b = \bar{c}$, $d = \bar{a}$, where bars indicate conjugate imaginaries. Then

$$T = \frac{az+\bar{c}}{cz+\bar{a}}.$$

Since the origin, 0, must transform into an interior point, and $T(0) = \bar{c}/\bar{a}$, we must have $|c| < |a|$. Hence the determinant $a\bar{a} - c\bar{c}$, which is real, must be positive. We shall insert such a positive factor in numerator and denominator that

$$a\bar{a} - c\bar{c} = 1.$$

There is no point congruent to 0 in a suitably small neighborhood of 0. In particular, 0 is not a fixed point for any transformation; hence $c \neq 0$, unless T be the identical transformation.

3. *Two Locus Problems.* Excluding the identical transformation, we shall solve the following two locus problems for the transformation T.

I. *Find the locus of a point in the neighborhood of which lengths and areas are unchanged in magnitude.*

Infinitesimal lengths are multiplied by $|T'(z)|$ and infinitesimal areas are multiplied by $|T'(z)|^2$. Since

$$T'(z) = \frac{1}{(cz+\bar{a})^2},$$

we have as the required locus the circle

$$C: \quad |cz+\bar{a}| = 1,$$

or

$$|z+\bar{a}/c| = 1/|c|.$$

This circle, which will be called the C-circle of the transformation T, has its center at the point $-\bar{a}/c$; its radius is $1/|c|$.

Writing the relation $a\bar{a}-c\bar{c}=1$ in the form

$$1+\frac{1}{|c|^2}=\left|\frac{\bar{a}}{c}\right|^2,$$

we see that the sum of the squares of the radii of K and C is equal to the square of the distance between their centers. Hence C is orthogonal to K. It follows that 0 is outside C.

If z is within C then

$$|z+\bar{a}/c|<1/|c|, \qquad |cz+\bar{a}|<1, \qquad |T'(z)|>1,$$

and lengths and areas in the neighborhood of z are increased in magnitude when transformed by T. Similarly, if z is outside C lengths and areas are decreased in magnitude when transformed by T.

II. *Find the locus of a point whose distance from* 0 *is unchanged.*

The required locus is

$$|z|=\left|\frac{az+\bar{c}}{cz+\bar{a}}\right|$$

or

$$|cz^2+\bar{a}z| = |az+\bar{c}|,$$

or

$$(cz^2+\bar{a}z)(\bar{c}\bar{z}^2+a\bar{z}) = (az+\bar{c})(\bar{a}\bar{z}+c).$$

On expanding and making use of the relation $c\bar{c}=a\bar{a}-1$ this factors into

$$\text{(a)} \qquad [z\bar{z}-1][(cz+\bar{a})(\bar{c}\bar{z}+a)-1]=0,$$

whence

$$|z|=1, \quad \text{or} \quad |cz+\bar{a}|=1.$$

The complete locus, then, consists of the two circles C and K.

If z is inside both C and K or outside both, the first member of (a) is greater than 0, and we find, on retracing our steps, that

$$|z|>\left|\frac{az+\bar{c}}{cz+\bar{a}}\right|.$$

The transform of z is nearer 0 than z is. Similarly, if z is inside one circle and outside the other the transform is farther from 0 than z is.

4. *Geometric Interpretation of* T. The inverse of T,

$$T^{-1} = \frac{-\bar{a}z + \bar{r}}{cz - a},$$

has the C-circle

$$C': \quad |z - a/c| = 1/|c|.$$

It is clear that T carries C into C'. For, T carries C into a circle C_0 without alteration of lengths; then T^{-1} transforms C_0 without alteration of lengths; whence C_0 coincides with C'.

The region exterior to both C and C' is transformed by T into the interior of C' and by T^{-1} into the interior of C. Let z be a point of the region, and let z be carried by T and T^{-1} into z' and z'' respectively. In both cases there is diminution of lengths and areas near z since z is exterior to both C-circles. Then the inverses, T^{-1} and T, carry z' and z'' respectively back to z with increase of lengths and areas. Hence z' is in C' and z'' is in C.

Let A, B and A', B' be the intersections of C and C' with K, the points being so designated that motion around C from A to B inside K is counter-clockwise about C, and motion from B' to A' around C' in K is counter-clockwise about C'. Now, on applying T, the interior arc AB of C is carried without alteration of length into either $A'B'$ or $B'A'$. The latter is impossible, for it is equivalent to a suitable rotation with 0 as fixed point, which is contrary to hypothesis. Hence A is transformed into A' and B into B'.

We can now give simple geometric interpretations of T. Let L be the perpendicular bisector of the line joining the centers of C and C'. L passes through 0. (In the special case that C and C' coincide let L be the line joining 0 to the center of C.) Either of the following pairs of inversions transforms the points of C exactly as T does and hence is identical with it:

(1) A reflection in L followed by an inversion in C';
(2) An inversion in C followed by a reflection in L.

We can show from these inversions that T is hyperbolic, elliptic, or parabolic according as C and C' are exterior to one another, intersect, or are tangent. The fixed points of the transformation are easily found geometrically.

5. *The Arrangement of the C-Circles.* We shall show first that there is an upper bound of the radii of the C-circles defined by the transformations of the group. There exists, by hypothesis, a circle of radius $\epsilon<1$ with 0 as center having no point congruent to 0 in its interior. We have then for any transformation of the group

$$|T(0)| = \left|\frac{\bar{c}}{\bar{a}}\right| \geqq \epsilon, \text{ whence } \left|-\frac{\bar{a}}{c}\right| \leqq \frac{1}{\epsilon};$$

that is, the distance from 0 to the center of C is not greater than $1/\epsilon$. Since 0 is outside C it follows that the radius of C is less than $1/\epsilon$.

Let

$$T_1 = \frac{\alpha z+\bar{\gamma}}{\gamma z+\bar{\alpha}}$$

be a second transformation of the group. Designate by C_1, C_1' the C-circles of T_1 and T_1^{-1}. Thus C_1 is $|z+\bar{\alpha}/\gamma| = 1/|\gamma|$. Let us now make the transformation

$$TT_1^{-1} = \frac{(-a\bar{\alpha}+\bar{c}\gamma)z+a\bar{\gamma}-\bar{c}\alpha}{(-c\bar{\alpha}+\bar{a}\gamma)z+c\bar{\gamma}-\bar{a}\alpha},$$

which, since $T_1 \neq T$, is not the identical transformation. Designating the radii of C, C_1, and the C-circle of TT_1^{-1} by r, r_1, r_2, respectively, we have

$$r_2 = \frac{1}{|-c\bar{\alpha}+\bar{a}\gamma|} = \frac{1}{|c\gamma|\cdot\left|-\frac{\bar{a}}{\gamma}-\left(-\frac{\bar{a}}{c}\right)\right|} = \frac{rr_1}{d},$$

where d is the distance between the centers of C and C_1. Since $r_2<1/\epsilon$ we have

$$d = \frac{rr_1}{r_2} > \epsilon r r_1.$$

Consider now all C-circles whose radii equal or exceed some positive number k. The distance between the centers of any two of the circles satisfies the inequality

$$d > \epsilon k^2.$$

Since the centers lie in the finite region bounded by K and by the circle of radius $1/\epsilon$ with center at 0 it follows that the number of such circles is finite.

It follows from the result just found that any closed region lying entirely within K, for example a circle with 0 as center and radius less than 1, is exterior to all but a finite number of C-circles.

Another consequence is that the transformations of the group are denumerable, since the radii of the corresponding C-circles are denumerable.

6. *The Fundamental region.* Let R be the region within K which lies outside all C-circles formed for the transformations of the group. The region is connected, since any of its points can be joined to 0 by a straight line segment which does not cross the boundary. We shall show that R is a fundamental region for the group; that is, (1) that no two interior points of R are congruent, and (2) that no region adjacent to R and lying in K can be added to R without the inclusion of points congruent to points of R.

The proof of the first property is immediate. The transform of any interior point of R by any transformation of the group, the identical transformation excepted, lies within some C-circle and hence is exterior to R.

To establish the second property we shall show first that if P, a point of C, the C-circle of some transformation T of the group, lies on the boundary of R, then P', the transform of P by T, also lies on the boundary of R. P' lies on C', the C-circle of T^{-1}.

Suppose P' does not lie on the boundary of R. Then P' lies within the C-circle of some transformation T_1 of the group. Consider the transformation T_1T. By the trans-

formation T lengths in the neighborhood of P are carried without alteration of magnitude into the neighborhood of P'. By T_1 lengths in the neighborhood of P' are magnified. Hence T_1T magnifies lengths in the neighborhood of P; consequently P is within the C-circle of T_1T. This is contrary to hypothesis; hence P' is on the boundary of R.

It follows from the preceding that if an arc ab of C forms part of the boundary of R the congruent arc $a'b'$ of C' is a part of the boundary. The transformation T^{-1} carries R into a region abutting R along ab. Any region adjacent to R abuts along some C-circle and contains points congruent by a suitable transformation to points of R. The second property is thus established.

The following properties of the region R are consequences of the preceding analysis:

1. R is bounded by arcs of circles orthogonal to K. The number of bounding arcs in a circle $|z| = r < 1$ is finite.

2. The bounding arcs are arranged in congruent pairs. Two congruent arcs of the boundary are equal in length, and congruent points thereof are equidistant from the center of K.

3. The vertices of a cycle (congruent vertices) lie on a circle concentric with K, since all are equidistant from the center of K. If the vertices of a cycle lie within K their number is finite.

4. R is the fundamental region of maximum area. For, a different fundamental region must contain points congruent to all points of R, and a shift of any part of R to a congruent position effects a diminution of area.

We shall now prove that *R and the regions congruent to it fill up without overlapping the whole interior of K.*

Suppose R_i and R_j, the transforms of R by T_i and T_j, overlap. Let z_1, z_2, z_3 be three points common to R_i and R_j. These are the transforms by T_i of three points z_1', z_2', z_3' of R, and the transforms by T_j of three points z_1'', z_2'', z_3'' of R. If $z_1' = z_1''$, $z_2' = z_2''$, $z_3' = z_3''$ then T_i and T_j are the same transformation, since they transform three points in the

same way, and R_i and R_j coincide. Otherwise the points of one pair, say z_1' and z_1'', are unequal. Being both congruent to z_1 they are congruent, which is impossible.

Let K_r be the circle $|z| = r < 1$. Let $a_1b_1, a_1'b_1', \cdots, a_nb_n, a_n'b_n'$ be the sides of R lying wholly or in part within K_r; and let T_i, $i = 1, 2, \cdots, n$, be the transformation carrying a_ib_i into $a_i'b_i'$. The transforms of all remaining sides of R are exterior to K_r, since the distance from 0 of any point on such a side is not decreased by any transformation of the group.

By applying $T_1, \cdots, T_n$ and their inverses we get regions congruent to R abutting on R along the sides $a_1b_1, \cdots, a_n'b_n'$. The sides of the new regions which lie in K_r are congruent to the sides just mentioned. By combinations of $T_1, \cdots, T_n$ and their inverses we can adjoin further regions along sides of these new regions, provided the sides lie in K_r; and the process can be continued as long as there are any free sides in K_r.

This process will end in a finite number of steps; for, each transformation carries R into the interior of a particular C-circle, and there is but a finite number of C-circles intersecting K_r. Hence K_r is covered by a finite number of regions. Since r may be chosen as near 1 as we like, it follows that the whole interior of K is covered.

We note from the preceding that all transformations of the group are formed by combinations of the transformations connecting congruent sides of R. These are therefore called generating transformations of the group. A further interesting fact is that a transformation which carries R into a region lying wholly or in part in a circle K_r concentric with K is a combination of those generating transformations only whose C-circles intersect K_r.

7. *An Important Special Case.* The sides of R may be finite or infinite in number. There are certain groups in which a fundamental region not extending to the principal circle is known to exist; for example, some of the groups

arising in connection with the uniformization problem.* For this case we have the following proposition.

If there exists a fundamental region F lying within $K_r(|z| = r < 1)$, then R lies within K_r, and the number of sides and of generating transformations is finite.

R and a finite number of its transforms, $R_1, \cdots, R_m$, will cover F completely. Carry the portion of F lying in each R_i into R by means of the transformation which carries R_i into R. The totality of these transforms of parts of F, together with the portion of F originally in R, fill up R completely. If this were not so we could construct a region D adjacent to one of these transformed regions and containing no points congruent to points of F. On carrying D back to the boundary of F we should have a region abutting on F and containing no points congruent to points of F, which is contrary to hypothesis.

Finally R is in K_r, for on transforming the parts of F into R the distance of no point from 0 is increased.

Since only a finite number of C-circles intersect K_r, it follows that R has a finite number of sides and that the number of generating transformations is finite.

THE RICE INSTITUTE

* See Osgood, *Lehrbuch der Funktionentheorie*, vol. I, 2d ed., pp. 721–25.

SOME PHASES OF DESCRIPTIVE GEOMETRY*

BY W. H. ROEVER

The purpose of this paper is to recall those phases of descriptive geometry which are involved in the construction of adequate pictures of space objects. These remarks are particularly addressed those who attempt to use pictures and thus admit a need for them. To quote from Klein, "Is it not as worthy an object of mathematics to be able to draw correctly as to be able correctly to calculate?"

The need for descriptive geometry was probably first felt by the artist and the architect. The former was interested in conveying by means of his drawings a clear notion of the spatial form of the object represented. The latter, on the other hand, made use of his process of drawing, not only to instruct the builder as to the form and size of the objects which he represented by his pictures, but, in addition, to solve by means of plane constructions the problems of space which were encountered by the mason and carpenter. Thus the art of stereotomy was developed in the Middle Ages. In separating the geometric constructions of this art from their application, Frézier, in 1738, laid the cornerstone of modern descriptive geometry. It was Gaspard Monge (1746 – 1818) however, who developed this new constructive geometry of space so as to elevate it to the dignity of a pure science to which he gave the name descriptive geometry.

At this point let us observe that the process of drawing can be performed only upon a surface, and, in particular. upon a plane surface. It thus becomes possible to execute graphically, i. e., with pencil, ruler and compasses (instruments of the geometer), the following fundamental operations of plane geometry (postulates of construction):

* An address delivered before the Southwestern Section of this Society at Ames, Iowa, November 29, 1924, by invitation of the program committee.

I 1. To determine the (straight) line connecting two points.
2. To find the point of intersection of two lines.
3. To construct a circle of given center and radius.

The corresponding fundamental operations of space are:

II 1. To find the line connecting two points of space.
2. To find the line of intersection of two planes.
3. To find the plane determined by a point and a line.
4. To find the point in which a line pierces a plane.
5. To construct a sphere of given center and radius.

There is, however, no process in space which is analogous to that of drawing in the plane. One might thus be tempted to say that the problems of space are incapable of graphical solution. In the art of stereotomy, solutions of problems of type II were virtually reduced to those of problems of type I, thus opening the way for the graphical solution of space problems.*

The principal purposes of descriptive geometry thus appear to be the following:

1. Representation of the objects of space by means of figures which lie in a plane (or upon a surface).

2. Solution of the problems of space by means of constructions which can be executed in the plane.

The criterion of what shall be meant by a good plane representative of a space object has, more or less unconsciously, been taken to be *that the plane representative when properly placed, shall produce upon the retinal surface of the eye an image which differs but little from that produced by the object itself.* Since the optical properties of the eye may be compared with those of the camera obscura, it is easy to see† that *the above criterion may be satisfied,* at least approximately, *by a plane picture which*

* See Gino Loria, *Vorlesungen über Darstellende Geometrie,* Leipzig, Teubner, 1907.

† See A. Schoenflies, *Einführung in die Hauptgesetze der Zeichnerischen Darstellungsmethoden,* Leipzig, Teubner, 1908.

is a central (or, by accommodation, also a parallel) *projection.* A photograph is such a plane picture.

In order that the solution of a space problem may be made possible by a construction in the plane, it is essential that there exist an *unambiguous correspondence between space and the plane.* To make this clear, we will observe that a single projection does not satisfy this requirement. For, while to each point of space P there corresponds a definite projection P' on the picture plane, it is *not* true, conversely, that to each point of the projection there is a definite point of space. Hence, in general, a single projection does not suffice if no information concerning the object projected is known. The reason that a single projection, like a photograph, may convey an adequate notion of an object (like a building) is because we have some information about the object, such, for instance, as the perpendicularity of some of its edges. If, however, such a relation concerning the object to be represented is not known, or not true, one projection of it is not sufficient to convey to the mind an adequate notion of its form. The projection of a cube might, for instance, also be the projection of any hexahedron of which the vertices are on the lines connecting the vertices of the cube with the center of projection. That two projections do, in general, suffice to set up an unambiguous correspondence between space and the plane, is illustrated by the stereoscope, which enables the user to combine the images of two pictures taken from different points of view and thus obtain the impression of solidity or relief.

Let us now illustrate several of the principal methods of descriptive geometry, showing how, in each of these methods, use is made either of *two projections* or of *one projection and information about the object projected.* For this purpose let us use, as an object to be represented, a solid of revolution obtained by revolving around one of its sides $O''Z''$ the part $O''D''E''F''Z''$ of a rectangle which remains after a corner has been removed by a circular cut $E''F''$ (see Fig. II). The solid of revolution

thus obtained is bounded by two planes (generated by $O''D''$ and $Z''F''$), by a circular cylinder (generated by $D''E''$) and by a torus (generated by the circle $E''F''$). Let us further suppose that points have been taken on the circle $E''F''$ in such a way that their (orthographic) projections on the line $O''Z''$ are equally spaced. In the revolution, these points generate circles of latitude of which the planes are perpendicular to the axis $O''Z''$ and are equally spaced.

Let us now choose several planes on which to project our solid. As a first plane of projection we will take the plane base π_1 (generated by $O''D''$) and as a second plane of projection, a plane π_2 parallel to the axis of revolution $O''Z''$. We will think of π_2 as coincident with the plane of our drawing, and of π_1 as intersecting π_2 in the ground line g_{12}. Then we will think of π_1 as rotated around g_{12} until it becomes coincident with π_2. The plane of the paper (π_2) is then a Mongean drawing plane. In it Fig. I is the orthographic projection of our solid on π_1 and Fig. II is that on π_2. Together these figures constitute a *Mongean representation* of our solid. It is easily seen that one of these projections alone is not sufficient, but that the two are sufficient. Thus we see that associated with a point of space P there are two points P', P'', in the drawing plane which lie on the same perpendicular to the ground line, and, conversely, to such a pair of points in the drawing plane there corresponds a point in space (namely, the point with which was associated this pair). From this it follows that associated with a line of space p there is a pair of lines p', p'' of the drawing plane; and conversely, in general, to a pair of lines in the drawing plane, there corresponds a line in space (namely, the line with which was associated this pair). A line of space p instead of being represented by its projections (p', p'') may also be represented by the points (P_1, P_2) (traces) in which it pierces the planes of projection π_1, and π_2 (see drawing). It is the extension of the latter method which is used to represent a plane. Thus a plane μ is represented by the lines (m_1, m_2), (traces)

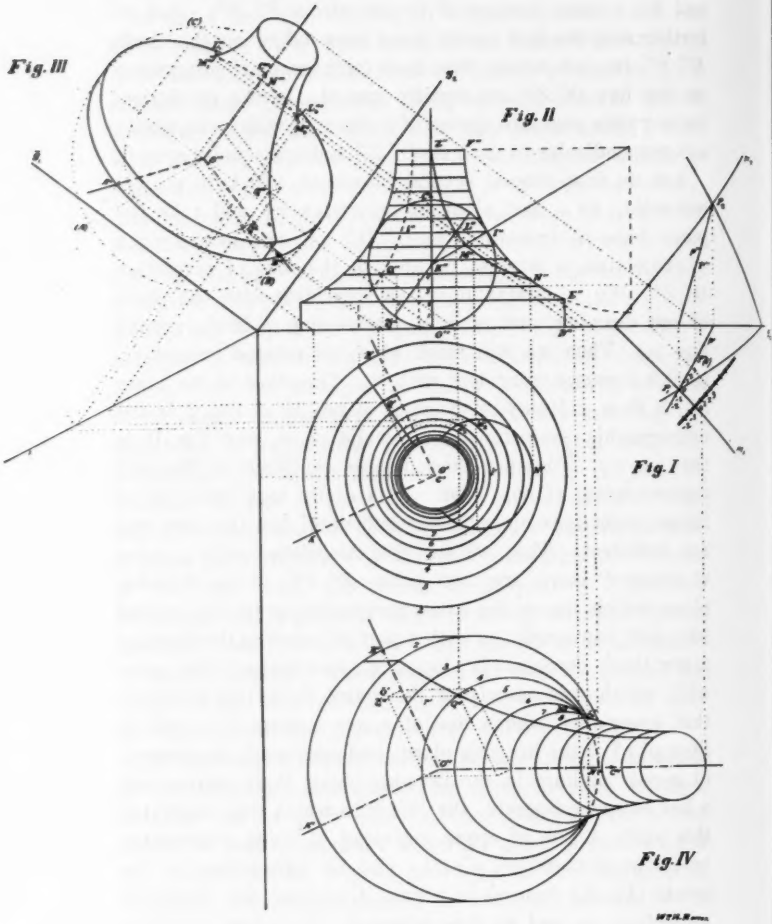
Fig. III
Fig. II
Fig. I
Fig. IV

in which it cuts the planes π_1 and π_2 (see drawing). Since these traces necessarily intersect on the ground line g_{12}, it follows that a general plane is adequately represented by two lines of the drawing plane which intersect on the ground line. It is thus evident that the Mongean method of representing space (by means of its elements, points, lines, planes) in the (drawing) plane, possesses the property of unambiguous correspondence.

A circle of latitude, such as that generated by the point I'', is represented by the circle i' in Fig. I and the straight line i'' in Fig. II. The corresponding representatives of other circles of latitude are also shown in these figures. These circles might be regarded as contour (level) lines if we now think of our solid as a mountain peak. If we attach to the circles in Fig. I numbers $(\cdots, 2, 3, 4, \cdots)$ representing the distances of these contour lines above the (datum) plane π_1, Fig. I becomes a *topographic representation* of our solid. In this method, a point is adequately represented by its (orthographic) projection and a number indicating its altitude. A (straight) line is represented by its projection and a scale, the intervals of which are the projections of the intervals into which the line in space is divided by equally spaced level planes; and finally, a plane is represented by its line of greatest slope. Thus, in the drawing, the point P' with the accompanying number (4) represents the point P (represented in the Mongean method by the pair of points P', P''), the line p' with the accompanying scale $(\cdots, -2, -1, 0, 1, 2, \cdots)$ represents the line p (represented in the Mongean method by the pair of lines p', p''), and the double line m' with the accompanying scale $(\cdots, -2, -1, 0, 1, 2, \cdots)$ represents the plane μ (represented in the Mongean method by the traces m_1, m_2). This method of representation also possesses the property of unambiguous correspondence.

Let us now find the (orthographic) projection (Fig. III of our solid on a plane $\pi \equiv [g_1, g_2]$ which is inclined to π_1, but perpendicular to π_2. The horizontal trace g_1 of

π will be perpendicular to g_{12}. We will then think of the plane π as rotated around its vertical trace g_2 (just as π_1 was rotated around g_{12}) until it comes into coincidence with the plane of the drawing π_2. By this rotation the trace g_1 assumes the new position $\bar{g}_1$, perpendicular to g_2 at the intersection of g_2 with g_{12}. The two figures, Fig. II and Fig. III, together constitute another Mongean representation of our solid, with g_2 as the corresponding ground line.

The points of Fig. III may be obtained from those of Figs. I and II by a knowledge of the following facts:

If Q denote a general point of space and Q', Q'', Q''' its projections on π_1, π_2, π, respectively, then just as Q' and Q'' lie on the same perpendicular to the ground line g_{12} so Q'' and Q''' lie on the same perpendicular to the ground line g_2. Furthermore, since the points Q' and Q''' are at equal distances from g_{12} and g_2 respectively, a line through Q' parallel to g_{12} meets a line trough Q''' parallel to g_2 on the bisector b of the angle formed by g_1 and $\bar{g}_1$ (see the drawing). Thus it is easy to obtain point by point the projection on π of the circular edges of our solid (generated by D'', E'' and F'').

In order to obtain the other bounding curve of Fig. III, let us observe that if we regard the rays which project our solid on the plane π as rays of light, this bounding curve is the projection on π of the *line of shade*, i. e., the curve which separates the illuminated portion of our solid from the unilluminated portion. Points of such a line of shade may be determined for a surface of revolution by the following well known construction. A surface of revolution may be regarded as the envelope of a one-parameter family of spheres of which the centers lie on the axis of revolution. Each sphere of this family has for its line of shade a great circle whose plane is perpendicular to the direction of the illuminating rays. On the other hand, each such sphere has contact with the envelope along a common circle of latitude of these two surfaces. The points in which these two circles intersect are points of the line of

shade of the envelope. To find the center of the sphere having contact with our surface along the circle of latitude generated by the point I'', we erect at I'' in the drawing plane a normal to the meridian curve $E''\,I''\,F''$. This cuts the axis $O''\,Z''$ in the required center K''. Through K'' we then draw a line parallel to g_2, since the rays of light are perpendicular to the plane π. This is the projection on π_2 of the line of shade of the sphere of center K, and cuts the corresponding projection i'' of the circle of contact in the point L''. This point is the vertical projection of two points of our line of shade. The horizontal projections of these points are found by drawing through L'' a line perpendicular to g_{12} cutting the circle i' in the points L_1' and L_2'. The corresponding projections L_1''', L_2''' on the plane π (Fig. III) are then found by the construction given in the preceding paragraph. Thus the bounding curves of Fig. III corresponding to the toroidal surface of our solid, may be found point by point. Apparently these curves stop abruptly at the points M_1''' and M_2'''. However, these points are not "points d'arrêt", but they are *cusps* of the complete curve of which the remaining portion $M_1'''\,N'''\,M_2'''$ is the bounding curve of the hidden (back) side of the torus. In fact this curve is a *parallel to the ellipse* into which the axial circle of the torus is projected on the plane π. For, the bounding curves of the (orthographic) projection of any tubular surface (i. e., envelope of a one-parameter family of spheres of constant radius) are parallels to the corresponding projection of the axial curve (i. e., locus of centers of enveloped spheres) of this surface, and the torus is a tubular surface with a circular axial curve.

With the same system of parallel projecting rays which were used to obtain the orthographic projection (Fig. III) of our solid on the plane π, let us also obtain the oblique projection (Fig. IV) of this solid on the plane π_1. In order that Fig. IV may not be superimposed on Fig. I we will drop the former down a little distance on the paper. Hence, now instead of using Q' as the horizontal projection of Q,

we will take $\bar{Q}'$ as this projection. Then, to obtain the required oblique projection Q^{IV} of the point Q we merely find the points in which the ray $r\,(r', r'')$ through the point $Q\,(\bar{Q}', Q'')$ pierces the plane π.

Since the edges, and other circles of latitude, of our solid are parallel to the plane of (oblique) projection, each of these circles projects into an equal circle and all that is needed is merely to find the (oblique) projections of the centers of such circles. The envelope of the circles thus obtained is the bounding curve of the (oblique) projection (Fig. IV) of our solid. But this may be otherwise obtained by merely finding (as for the point Q) projections of points L of the line of shade.

If in Fig. IV (as in Fig. I) we attach numbers to the circles indicating the altitudes of the planes of the circular sections of which these circles are the (oblique) projection, Fig. IV alone is an adequate plane representation of our solid. This representation has the advantage over that given by Fig. I with the corresponding attached numbers, in that it conveys with less effort, on the part of an observer, a notion of the spatial form represented.

In a similar manner, Fig. III with numbered ellipses (not drawn) into which the circles of latitude of our solid project (orthographically), serves as an adequate plane representation of our solid.

However, instead of drawing these ellipses in Fig. III, or the circles in Fig. IV, we might merely give the scale ($O'''\,Z'''$ in Fig. III, $O^{IV}\,Z^{IV}$ in Fig. IV) into which the vertical scale $O\,Z$ (shown in true size in Fig. I) projects. In addition to this scaled axis ($O'''\,Z'''$ in Fig. III or $O^{IV}\,Z^{IV}$ in Fig. IV) let us now take two other scaled axes ($O'''\,X'''$, $O'''\,Y'''$ in Fig. III or $O^{IV}\,X^{IV}$, $O^{IV}\,Y^{IV}$ in Fig. IV), these being the projections of space axes $O\,X$ and $O\,Y$, each bearing the scale of $O\,Z$ and perpendicular to each other and to $O\,Z$. In addition to the projection of the general point Q (Q''' in Fig. III or Q^{IV} in Fig. IV), let us find that of the orthographic projection $\bar{Q}$ of the point Q on the

plane π_1 ($\tilde{Q}'''$ in Fig. III or $\tilde{Q}^{IV}$ in Fig. IV). Then this pair of points (Q''', $\tilde{Q}'''$ in Fig. III or Q^{IV} $\tilde{Q}^{IV}$ in Fig. IV) is sufficient to represent adequately a point of space Q. For we can find from the three axonometric axes ($O'''X'''$, $O'''Y'''$, $O'''Z'''$ in Fig. III or $O^{IV}X^{IV}$, $O^{IV}Y^{IV}$, $O^{IV}Z^{IV}$ in Fig. IV) the coordinates of the point Q in space.

We thus have the *axonometric method* of representation (orthographic in Fig. III, oblique in Fig. IV), in which two points on a line parallel to one of the axonometric axes represent adequately a point of space. Likewise, in general, two lines of the picture plane represent a line of space, and, finally, a plane of space is adequately represented by two lines of the picture plane which intersect on one of the axonometric axes.

While Figs. III and IV were obtained by Mongean processes from the Mongean representation consisting of Figs. I and II, they may be obtained directly by the axonometric method. In the use of this method, one must be able to determine the scales when the directions of the axes are given. The relationship which is involved between these angles and scales is, in the case of orthographic axonometry, expressed by the following theorem.

THEOREM. *Any two conjugate diameters and the minor axis of an ellipse may be regarded, as far as directions are concerned, as the orthographic projection on the plane of the ellipse of three mutually perpendicular axes; if these axes be taken as cartesian axes and the plane of the ellipse as the picture plane, the ratios borne by the halves of the chosen conjugate diameters and the focal distance of the ellipse to the half major axis of the ellipse are the foreshortening ratios,* i. e., *the numbers by which the scales on the space axes must be multiplied in order to obtain those on the axonometric axes.* This relationship is indicated in Fig. III, in which the ratios of $O'''A'''$, $O'''B'''$, $O'''C'''$ to $O'''(A)$ are the foreshortening ratios.

In the case of oblique axonometry, however, one may choose at pleasure both the angles and the scales. The

truth of this assertion is contained in the following famous theorem, which is due to Pohlke.

POHLKE'S THEOREM. *Three straight-line segments of arbitrary lengths in a plane, drawn from a point and making arbitrary angles with each other, can be regarded as a parallel projection of three equal segments drawn from the origin on three rectangular coordinate axes; however, not more than one of the given segments or one of the given angles can vanish*.*

Pictures of the type of Fig. III and Fig. IV make more of an appeal than do those of the type of Figs. I and II. In the case of most of the objects of architecture and technology there are among the bounding surfaces certain mutually perpendicular planes. These planes, or parallel planes, are usually selected as the Mongean planes of representation. Thus certain plane faces appear merely as lines, and hence these projections do not convey as easily a notion of the spatial form of the objects as do projections on planes not parallel to any of the mutually perpendicular planes. Hence, here also, pictures of the type of Fig. III and Fig. IV, which can easily be made directly by the axonometric method, prove very desirable.

All of the methods described in this paper possess the property of unambiguous correspondence. In each is given the plane representative of the space elements: point, line, plane. The first four of the space operations II thus have their counterparts in the picture plane and hence all of the three-dimensional problems of pure geometry of position can be solved in the plane by each of these methods. The fifth of the operations II can, in like manner, be replaced by the third of operations I, and thus also is made possible a graphical solution of all perpendicularity and metrical problems of space by means of constructions in the plane.

WASHINGTON UNIVERSITY, ST. LOUIS.

* For a collection of proofs of this theorem see Wendling, *Der Fundamentalsatz der Axonometrie*, Zürich, Speidel, 1912.

THE FIRST CARUS MONOGRAPH

Calculus of Variations. By Gilbert Ames Bliss. Carus Mathematical Monographs, No. 1. Published for the Mathematical Association of America by the Open Court Publishing Company, Chicago, 1925. 13+189 pp.

One of the important concepts introduced into mathematical thought by E. H. Moore is that of the "Extensional Attainability" of properties (see New Haven Colloquium Lectures, p. 53). This concept, apart from its technical significance in the theory of properties of classes, admits of another interpretation of perhaps broader applicability, one suggestive of conquest. So, an infant reaching out for its playthings might be said to be experimenting with the extensional attainability of satisfaction for its desires; explorers illustrate the extensional attainability of man's control over the globe; other instances will occur to the reader.

The Carus Mathematical Monographs, of which Professor Bliss's book is the first, are intended "to contribute to the dissemination of mathematical knowledge by making accessible at nominal cost a series of expository presentations of the best thoughts and keenest researches in pure and applied mathematics," "in a manner comprehensible not only to teachers and students specializing in mathematics, but also to scientific workers in other fields, and especially to the wide circle of thoughtful people who, having a moderate acquaintance with elementary mathematics, wish to extend their knowledge without prolonged and critical study of the mathematical journals and treatises." Is this not an exhibition of faith in the extensional attainability of a mathematically informed public? It certainly is most fitting that this series of monographs should have been conceived by a Chicago group and that its first number should come from the pen of one of the members of the Department of Mathematics at the University of Chicago.

To Mrs. Mary Hegeler Carus and to her son Dr. Edward Carus belongs the honor of having recognized the importance of such an undertaking and of having provided the necessary means. Through the publication of these books, the Open Court Publishing Company continues its fine service to mathematical education in this country.

To Professor H. E. Slaught belongs the credit for the inception of the idea which gave rise to the monographs and for having solicitously guided it to successful realization. This series of books will forever be a reminder of his farsighted and intelligent devotion to the cause of mathematical education and to his skill in leading it on into new and significant fields of conquest.

In how far the wider dissemination of knowledge contributes to the

enlargement of its domain, not many would venture to say. And, whether mathematics is better served by making results long familiar to the specialists accessible to a wider group than by the publication of new results, is, or should be, a rather futile question. Neither of these tasks should be allowed to be neglected. We must constantly labor to enlarge the foundations if they are to support the ceaselessly expanding superstructure, without, however, using up so much material in the process that the superstructure must suffer.

The simile is manifestly inadequate because it does not allow for the human elements involved. Diffusion of ideas among a larger group is likely to generate curiosity, to lead to the discovery of hitherto unsuspected connections, to suggest new ideas. There is no doubt that a much more healthy development may be expected if broader connecting avenues are laid out between the fields of pure and applied mathematics, if the engineers and the physicists became acquainted with some of the developments and some of the results that have been secured from ideas that perhaps were first brought to light in their fields of knowledge. If the Carus monographs contribute to a removal of the barriers which keep modern mathematical science isolated, they will fully justify the faith of the founders and the hopes of its promoters. May the series prove to be a multiple series stretching out its arms in many directions, so as to attain the widest possible extension.

In accordance with the general aim set for the series, the volume now under review has the purpose of bringing the general methods of the calculus of variations within the reach of a larger public than can be expected to master all the requirements which a systematic study of the subject would require. And to the reviewer it seems that the book succeeds admirably in this purpose. Whether this judgment is correct can only be estimated by the future historian.

The process is inductive. In an introductory chapter of sixteen pages. the calculus of variations is presented to the reader in a semihistorical way, chiefly by means of an analogy with the problem of finding the maxima and minima of functions, and through some of its illustrative problems. Three of these, all belonging to the "simplest problem of the calculus of variations," viz. the problems of the shortest distance, of the brachistochrone, and of the surface of revolution of minimum area, form the subject matter of chapters II, III, and IV, respectively, a total of 111 pages. The fifth chapter treats the minimizing of the integral $\int f(x, y, y')\,dx$. Then follow a list of references, notes, and index.

"The author assumes," so the warning on the jacket reads, "that the reader has an acquaintance with the elementary principles of the Differential and Integral Calculus." Even with this assumption as to the amount of preparation of his readers, the author must have been in doubt many times as to whether or not to presuppose know-

ledge of a particular fact. In some instances he apparently concluded that a reminder was the only thing necessary. As such we have to consider for instance the clause on page 21 concerning the derivative of an integral with respect to its upper limit, and the brief paragraph on the cycloid on page 52. Would not references to fuller treatment of such questions be useful in these places?

Each of the problems to which separate chapters are devoted receives a more complete treatment than they do when used as examples following the development of the general theory, a treatment moreover which utilizes to the full the more recent work done in connection with the classical theory. In preparation for the derivation of the Euler equation in the shortest distance problem, the lemma, usually attributed to Du Bois Reymond, is proved in a very simple manner, although perhaps a little too cleverly for the unsophiscated reader. The Hilbert independent integral, the concept of a field and related ideas and theorems are introduced in each of chapters II, III, IV, in the special forms which they take for the problems there treated. The usual methods of the sufficiency proofs are exemplified in a similar manner, as are also the cases in which one or two endpoints are variable on arbitrary curves. In the discussion of these latter questions the author is not satisfied to treat merely first order conditions, but gives a complete account of the theory of the focal point, which receives in each of the special cases a more or less simple geometrical interpretation. The proof of the existence of a unique extremal through two given points and the construction of a field in the brachistochrone problem are carried out by drawing in a very skillful way upon well known properties of the cycloid. In connection with the same problem we are introduced to the envelope theorem and to the geometric proof of Jacobi's theorem which depends upon it.

Particularly valuable is the presentation in the fourth chapter of the results obtained by Sinclair and MacNeish in the catenary problem, not heretofore available in a connected form, and the clear discussion of the relation between the catenary and the Goldschmidt straight line solutions of the problem of the surface of revolution of minimum area.

The final chapter, more than any of the others, is of the usual textbook character. The reader who has followed the discussions of the three special problems in the preceding chapters should be well prepared to understand now the general theory as here presented. He meets again, in the form of general theorems, statements with whose general character he has had an opportunity to become acquainted. But the chapter contains more than these generalizations of the results previously obtained for special cases. It is here that we learn for the first time of the distinction between weak and strong minima, of the Weierstrass E-function, of Legendre's condition, of the Weierstrass-

Erdmann corner point condition, etc., which for various reasons have not been brought forward before. It is furthermore made clear that the geometrical treatment of the conjugate point condition is not sufficient in all cases. This leads to a discussion of the second variation by the elegant method which the author introduced some years ago and which has since been applied by several of his pupils to a variety of more general problems, but a systematic exposition of which has not hitherto been available outside the journals. Moreover, the reader who merely wants to get the results will find an accurate statement of sets of necessary and of sufficient conditions for the cases both of fixed and of variable endpoints. The chapter closes in the same key in which the first one opened, viz., with historical remarks.

Enough has been said to make clear that in the judgment of the reviewer, Professor Bliss has carried through a difficult task with remarkable success. We have here a new type of mathematical book. It is not a textbook, neither is it addressed to the specialist, actual or *in spe*. It is intended for that very indefinite group, the intelligent general public. This group surely includes teachers of mathematics, even those who may have taken a course in the calculus of variations! Certainly engineers who expect to do more than routine work, physicists and chemists and other workers in natural science can profit largely from familiarizing themselves with the leading ideas set forth in this book. To many it should be of immediate usefulness.

But to write on an advanced mathematical topic for a public of such mixed and uncertain qualifications is much like lecturing to a radio audience: one never can tell whether there is any real catching on. In his selection of material the author had to trust to his judgment; the style of treatment is in a large measure determined by this choice. Now one can readily conceive of other selections which might have been made. A treatment of an isoperimetric problem, mention at least of some problems leading to integrals with more unknown functions, and to extrema of multiple integrals would seem to me to be necessary in order to give an adequate picture of the broad scope of the calculus of variations. This remark however amounts to little else than to asking for "more of the same kind"; for I should not want any of these topics to replace anything that is in the book now. And this indeed is my chief criticism; the book is too short. It is a fine beginning; but it should be continued.

This then is the remark with which I wish to close this review: Let the series of Carus Monographs, for which I anticipate a long and successful career, develop a semi-periodic character which will make it possible to carry forward to other parts of the field the discussion of the calculus of variations which Professor Bliss's volume has so excellently initiated.

ARNOLD DRESDEN

AUERBACH ON MATHEMATICAL PHYSICS

Die Methoden der theoretischen Physik. By Felix Auerbach. Leipzig, Akademische Verlagsgesellschaft, 1925. x + 436 pp.

The book to be reviewed is written by a physicist for students of physics and is meant as an introduction to theoretical, i. e. mathematical physics. The author's program is to present the underlying notions and principles as well as to give a compendium of the mathematical devices which the student is likely to need in a more profound study of the subject. The following extract from the table of contents will show that the ground covered by the author is very extensive:

I. Notions and principles, 27 pp. II. Formal methods, 14 pp.* III. Elementary equations, 26 pp. IV. Difference, differential, and integral calculus, 57 pp. V. Ordinary differential equations, 56 pp. VI. Partial differential equations, 125 pp. VII. Integral and functional equations, 23 pp. VIII. Theory of molecules; Statistical methods, 47 pp. IX. Geometrical, graphical, and vector methods, 51 pp. X. Concluding remarks, 6 pp. Index, 4 pp.

It is rather obvious that the task of treating such a wide range of ideas in an adequate manner is a very trying one. The best parts of the book are those where the author is concerned with the applications of some special theory to particular problems. In fairness it must be said that such problems occupy the greater portion of the book. The examples are usually well chosen and the calculations are carried out in quite an elegant manner. But every now and then we encounter a piece of a general mathematical theory which is presented in an awkward and doubtful manner. The reviewer is perfectly aware of the fact that mathematical rigor can not and perhaps should not be the primary concern of the physicist. But in a book which is meant to introduce a young student into the realm of theoretical physics where he is going to use mathematical tools in all his work, it is only fair to the student to tell him something about the limitations of these tools as instruments of research. This of course applies essentially to questions involving some kind of infinite process or limiting passage where the student needs a timely warning. The author's habit of stating results without mentioning the conditions under which these results hold, is deplorable; it makes the value of the information imparted illusory, and it is scarcely conducive to sound mental habits

* *Wortsprachliche Methodik.* What is meant is methods based upon reduction to general principles by means of qualitative discussion without the use of calculations.

in the student. Occasionally the statements are wrong or at least misleading.

Examples of such failings are plentiful; the theory of series supplies several. Here are a few samples. On page 95 we are told that in order that the series Σu_n shall be convergent, the limit of u_{n+1}/u_n must be less than unity for sufficiently large values of n. The continuation takes care of the case when the limit is unity but adds other deep mysteries. On page 98 we are told that a series with alternating terms is always convergent if the absolute values (Betrag) of the terms converge to zero with increasing values of n. The treatment of power series on the same page is not very satisfactory: the series $\Sigma a_n x^n$ converges when x lies between $-1/\beta$ and $+1/\beta$ where β is the limit of a_{n+1}/a_n. Apply the rule to the case $a_n = \sin n\alpha$, please! On the next page the author says that for values of x which are greater than one it is better to expand functions according to descending powers of x, because of considerations of convergence. Such considerations do not bother the author on page 278. Here he wants to justify the restriction of the parameter in Legendre's equation to integral values as follows: Wenn nun n keine ganze Zahl ist, gehen die Reihen ohne Ende fort, und für $\mu = 0$ werden beide Reihen selbst unendlich, was nicht sein darf. These series are power series in $1/\mu$ and only convergent for $|\mu| > 1$.

Only five pages of the seventh chapter deal with integral equations in the technical sense of the word. Evidently our author does not like these equations; he begins by saying that they have not completely come up to expectations, and ends the discussion by remarking that the use of integral equations has not proved to be a great positive gain for physics since in all cases which cannot be solved by other means, the application of integral equations usually encounters difficulties or is wanting in lucidity.

It is hard to estimate the value of the book for the particular class of readers to which it is directed, and it is scarcely fair for a mathematician to judge in this matter. The book does give a great amount of useful information. As to the misinformation, perhaps the habit of slurring over everything abstract that characterizes the average young student will preserve the purity of his mind until he reaches the years of discretion.

EINAR HILLE

SHORTER NOTICES

Curve Sgembe Speciali Algebriche e Transcendenti. Volume primo: Curve algebriche. By Gino Loria. Bologna, N. Zanichelli, 1925. x+374 pp.

The monumental treatise on plane curves* by the same author has now been followed by the first volume of an extensive work of a similar nature on curves of three-dimensional space. The first chapter contains a fairly full derivation of the ordinary differential theory of space curves in general, followed by another on curves derived from a given one, such as radial curves, evolutes, filar and planar, pedals, parallel curves, medians of two given curves, etc. Most of these properties are not made further use of, and for. This reason it is curious to find them in a book with this title.

The subject of the book proper begins with a short chapter on the conic in space, treated parametrically, followed by one hundred and thirty pages on the cubic; first the general projective theory, then a detailed consideration of each metric type. The treatment is very elementary and complete. Then come the elliptic and the rational quartics, and finally a brief outline of quintics, and scarcely more than a mention of certain sextics and a few other isolated curves.

While the text is well supplied with historical and bibliographical notes, it is by no means exhaustive; it does include a number of quite recent papers, including some from our BULLETIN and TRANSACTIONS.

The characteristics of space algebraic curves, analogous to Plücker's numbers for the plane, are not given; in several cases rather primitive methods of deriving some of them are employed, when they could be read out of tables uniquely from others already known.

The book is not provided with an index, either of names or of subject matter. In a work of this kind, such indexes are particularly useful. The matter is so arranged, however, that a topic can be found without undue effort.

The mechanical execution is excellent, and the type large and clear. On the whole but very few typographical errors were noticed, and of these none would be liable to confuse the reader. The citation of German titles suffers most heavily in this regard.

VIRGIL SNYDER

---

* *Spezielle algebraische und transcendente ebene Curven, Theorie und Geschichte, Autorisierte, nach dem italienischen Manuscript bearbeitete deutsche Ausgabe von Fritz Schütte*, 1902; 2nd edition in two volumes, 1910–1911.

L'Analysis Situs et la Géométrie Algébrique. By S. Lefschetz. Paris, Gauthier-Villars, 1924.

As a condensed account of the theory of algebraic manifolds, this little volume is so successful that it would seem beside the point to criticise the somewhat hasty manner in which the author disposes of several rather delicate details. Professor Lefschetz has not only given the theory a new unity, but has clarified and extended it at more than one point by the systematic use of topological methods.

First, we are given a brief résumé dealing with the theory of cycles on an algebraic surface, the connectivity numbers and coefficients of torsion, the intersection numbers associated with cycles intersecting in points. Skilful use is made throughout the book of the simple theorem that if one of two cycles intersecting in points is a bounding cycle, the algebraic sum of the points of intersection is zero. It may be mentioned here, in passing, that there are a number of topological invariants associated with the theory of cycles that meet, not in points, but in curves, surfaces, etc. So far as the reviewer knows, the significance of these invariants in the theory of algebraic manifolds has never been studied.

There follows a discussion of the topological characteristics of an algebraic surface, during the course of which a number of the fundamental constants of algebraic geometry reappear as topological invariants. The surface is thought of as swept out by its intersection with a plane varying in a pencil of planes. A general plane section of the surface is a curve C of genus p on which $2p$ 1-cycles, $\gamma_1, \gamma_2, \cdots, \gamma_{2p}$, independent with respect to C, may be traced. If the plane is made to vary in its pencil so as, ultimately, to return to its original position, the cycles γ_i undergo a linear transformation. It would be interesting to see how far the theory of algebraic manifolds could be carried without making use of anything more than the group of linear transformations engendered on the cycles γ_i. Along these lines, it might be possible to develop a perfectly general theory of surfaces without going into the question of the resolution of singular points and curves.

There is an important theorem due to Lefschetz himself to the effect that a cycle is algebraic if and only if no double integrals of the first kind have periods with respect to it. This and other considerations suggest that, in the theory of algebraic manifolds, it might be advisable to modify the definition of an homology so as to make it read as follows: A cycle C is homologous to zero, $C \sim 0$, if, and only if, it is bounding *or algebraic.* The invariants (Betti numbers, coefficients of torsion, etc.) that would follow from this definition of an homology would have the advantage of being unaltered under birational transformations, whereas the strictly topological invariants that follow from the ordinary definition do not have this property.

Perhaps the most illuminating chapter in the book is the one dealing with systems of curves on a surface. Here we find a fundamental theorem that two curves are algebraically equivalent if, and only if, they are homologous in the ordinary sense of analysis situs. Another interesting fact, as pointed out by Lefschetz, is that the so-called "virtual" curves, which have a purely symbolic existence from the algebraic point of view, may be interpreted in topological terms as non-algebraic cycles of the surface. The book also contains other interesting chapters on algebraic manifolds of higher dimensions and on abelian functions, not to mention two notes in the form of postscripts.

J. W. ALEXANDER

Sophus Lie's Gesammelte Abhandlungen (Samlede Avhandlinger). Edited by Friedrich Engel and Poul Heegaard. Volume V: *Abhandlungen über die Theorie der Transformationsgruppen,* erste Abteilung (*Avhandlinger om Transformationsgruppernes Teori,* første Avdeling), edited by Friedrich Engel. Leipzig, B. G. Teubner, and Kristiania, H. Aschehoug and Co., 1924. xii+776 pages.

The fifth volume of Lie's collected memoirs is the second of the series to be published; it is preceded by the third volume which was published in 1922 (and was reviewed in this BULLETIN, vol. 29 (1923), pp. 367–369). The published memoirs of Lie are to be gathered together in six volumes while a seventh is to be devoted to the principal works found among his literary remains. The memoirs on geometry are to go into volumes I and II; those on differential equations into volumes III and IV; and those on transformation groups into volumes V and VI. Naturally these divisions are not rigorously separated from each other; and each volume will contain memoirs belonging in part to all three divisions. In each of the three divisions the arrangement is chronological; and, generally, the material in the first of any two related volumes is that which was first published at Christiania, while the second of these volumes is devoted principally to Memoirs from the MATHEMATISCHE ANNALEN and the publications of the Leipzig Akademie. In this way it is brought about that no volume will contain two memoirs one of which is mainly a reworking of the other.

As the subtitle indicates, the fifth volume is given to memoirs on transformation groups. It contains the earlier of them, up to February 1889. Of the 560 pages required to print the included memoirs, about 320 pages are given to those in which "Transformationsgruppen" is the main word in the title; a large part of the remaining space is given to memoirs dealing with the applications of the theory of transformation groups to differential equations. The notes from the Norwegian (none of them long) are translated into German, so that the whole body of this volume appears in German. On pages 669–673 of the notes an

address of Lie's is published in French, the language in which it was spoken. (In volume III the French articles were published in French, all others appearing in German.) Some verbal changes have again been made in the articles on reprinting; and a list of these is given on pages 561–582.

The notes and supplementary matter in volume III made up more than a third of the whole. The editor stated in that volume that volume V would require much fewer notes; but the event has shown that he was mistaken, the space given to supplementary matter in volume V being only a little less than in volume III. On pages 583–614 we have an account of the beginnings of the theory of transformation groups as set forth by Lie in his letters to Adolph Mayer in 1873 and 1874. Pages 615–755 are given to the editor's notes on the separate memoirs (and include the address already mentioned). These notes contain numerous matters of detail, explanatory remarks, suggestions and cross references, together with a few discussions of considerable length (such as that on pages 643-668). An extensive index to the volume is given on pages 756–774.

The editorial work (here as in volume III) is marked throughout with evidence of that care and patience which belong only to a labor of love. It is done in such way as to render great service to all those who will have occasion to use the memoirs of Lie which are here reproduced. The next volume of the series which is to be printed is the sixth, and in the preface it is indicated that work upon it will begin immediately. We can not hope more for it than that it will be edited and printed with the same care as volumes III and V.

R. D. CARMICHAEL

Johannes Kepler. Mysterium Cosmographicum. Das Weltgeheimnis. Uebersetzt und eingeleitet von Max Caspar. Augsburg, Dr. Benno Filser Verlag, 1923. xxxi+150 pp.

The large increase in the number of students of mathematics and natural science who are unfamiliar with the Latin language makes it more and more useful and desirable to have the greatest works of the pioneers of modern science translated into modern tongues. Among these pioneers Kepler deserves a distinguished place, first because of his work in geometry, in which he prepared the way, through his "infinitesimal method," for the invention of integral calculus; and secondly because in astronomy he laid the foundation for the modern view of the solar system, through his famous three "laws."

Thus there is a real place in the literature of the history of science for such works as the one under review. The *Mysterium Cosmographicum* was Kepler's first work, and it attracted sufficient notice

to assure him of a permanent career as an astronomer. The translation of this work is preceded by an interesting and valuable Introduction, in which Dr. Caspar gives a discussion of Kepler's place in the history of science, of his literary style and of the relation of the *Mysterium Cosmographicum* to his other works. It is made clear that in this fantastic-appearing mixture of science, pseudo-science, and mysticism are found the germs, and more than the germs, of the great discoveries of the author's later years.

Kepler's discussion of his theme, in spite of the fact that it is in places labored and tedious, has yet a frankness and naiveté that give it interest and charm. And the light which this work throws upon the beginnings of modern astronomy is alone sufficient to repay the time spent in reading it. Written but fifty years after Copernicus's *De Revolutionibus Orbium* it very naturally begins with a discussion of the relative merits of the Ptolemaic and Copernican systems, in which the greater simplicity of the latter is the chief argument which Kepler stresses. He then proceeds to explain his view of the orbits and motions of the planets. He makes the distances of the planets from the sun equal to the radii of the inscribed and circumscribed spheres of a series of regular polyhedra. Mercury's orbit is placed inside an octahedron, Venus outside it and inside an icosahedron; outside this, and inside a dodecahedron, comes the earth; Mars is then outside this and inside a tetrahedron; Jupiter outside this and inside a cube, while lastly the orbit of Saturn is drawn about the cube. Kepler gives ingenious reasons why the order of the solids should be just this and no other Thus, the cube must be the outermost, "since the number of sides of the cube, six, is a perfect number." After detailed statements of the numerical relations that exist among the sides and radii of the five solids, he attempts to show that the relative distances of the planets are in agreement with these computations; and while he naturally finds discrepancies of various magnitudes, he has a ready excuse for each, after considering which he is able to convince himself that he has found the true explanation of the relative distances of the planets, as well as of the "fact" that there are exactly six.

The translation has been well done, to judge from numerous passages which the reviewer chose at random and compared. The figures are not well printed, the one on page 89 being actually incorrect; but otherwise there are few misprints.

It would be a valuable contribution to the history of science if we could have a number of translations into English such as this work and the Ostwald "Klassiker" give to the German student. Not to go farther, why can we not have an accessible English translation of Newton's most important works?

R. B. McClenon

Oeuvres de G. H. Halphen. Publiées par les soins de C. Jordan, H. Poincaré, É. Picard, avec la collaboration de E. Vessiot. Tome IV.* Paris, Gauthier-Villars, 1924. xv+657 pp.

The works of G. H. Halphen with the exception of the three-volume *Traité des Fonctions Elliptiques,* are contained in four volumes the last of which is now before us. Each volume opens with a "notice" concerning Halphen and his works written either on the occasion of his death or not long afterwards; in fact, the first volume contains two such notices. It has also Halphen's own account (pages 1–47) of his mathematical works written on the occasion of his candidature for membership in the Academy of Sciences. It is natural to find at the beginning of his *Oeuvres* this summary of his own works up to 1885. Except for this paper his memoirs are arranged in the *Oeuvres* in almost chronological order, the deviations from this order being unimportant and not many in number. Owing to Halphen's custom of carrying one investigation to a complete achievement before taking up another and owing to the fact that the order of completion of memoirs is also very nearly their order of publication, the chronological arrangement of Halphen's work is also to a considerable extent a logical arrangement. But the former in some respects departs considerably from the latter so that it is desirable to have added (as on pages 645–653) the "Table Méthodique Générale."

Among the memoirs contained in the first volume are those on the characteristics in the theory of conics, on singular points of plane curves, and on the points where an algebraic curve satisfies a condition expressed by means of a given differential equation. For the second volume we may mention those on the singularities of twisted curves and of algebraic surfaces, on the differential invariants of plane and twisted curves, and on the representation of functions in various types of series. Two of the four memoirs in volume III contain the best work of Halphen. These are the memoir on the reduction of linear differential equations to integrable forms and that on the classification of twisted algebraic curves.

This leaves to the fourth volume the last works of Halphen other than his treatise on elliptic functions, the memoirs and papers appearing in 1883 to 1890, together which such unpublished material as his editors have seen fit to preserve with his collected works. The papers for the most part deal with differential equations, the theory of numbers, and elliptic functions and their applications; they are more fragmentary in character than the comprehensive memoirs in the earlier volumes, doubtless owing to the fact that Halphen's consecutive

* The previous volumes have been reviewed in this BULLETIN, vol. 27 1920-21), pp. 466–468 and vol. 28 (1921-1922), pp. 271–272.

thought in his last years was being given largely to his treatise on elliptic functions. Here is reproduced also his work on the theory of the singularities of plane curves, originally published as an appendix to the French translation of Salmon's analytical geometry.

In addition to these previously published writings there are two appendices. The "Mémoires et Fragments Inédits" (pages 465–627) contain a considerable variety of material taken from the manuscripts which Halphen left and thus preserve what is most useful in this material. The editors desired also to publish the letters which Halphen had written to several scientists with whom he had been in correspondence; but they were able to bring together only a certain number of extracts from the letters of a single one of his correspondents, namely, Zeuthen. The "Extraits de Lettres à Zeuthen" (pages 628–644) were chosen and transcribed by Zeuthen himself. These letters belong to the period 1876 to 1881 of Halphen's life.

Owing to the fact that the various "notices" in the four volumes contain several brief accounts of Halphen's work the reviewer is relieved of the duty of making an analysis of the separate memoirs.

R. D. CARMICHAEL

The Mathematical Theory of Relativity. 2d edition. By A. S. Eddington. Cambridge, University Press, 1924. ix + 270 pages.

The second edition of this important work differs from the first mainly in the addition of thirteen pages (pp. 241–263) of "Supplementary Notes" relating to various sections throughout the whole volume. There is no general revision of the main body of the text. A few corrections have been made, and in such way as not to disturb the paging. An erroneous formula [numbered (59.6)] on page 137 has been suppressed, the paragraph containing it having been replaced by another. The remarks on the problem of the homogeneous sphere on page 170 have been modified, this page having been largely rewritten. The other changes are less important than these two.

R. D. CARMICHAEL

Statistical Methods. By Frederick Cecil Mills. New York, Henry Holt and Co., 1924. xvi+604 pp.

This book has been prepared with particular "reference to the specific needs of quantitative workers in economics and business" and therefore calls for little comment here. It includes, in addition to an extensive elementary treatment of statistical graphs, averages, correlation theory, etc., an extensive treatment of index numbers and time series. It has been carefully prepared and with suitable abbreviations could be used as a textbook in a mathematical course in statistics where no collegiate course in mathematics is a prerequisite.

C. H. FORSYTH

Petit Traité de Perspective. By Raoul Bricard. Paris, Librairie Vuibert, 1924. 87 pp.

This little treatise of perspective contains a partial reproduction of the course given by Bricard at the Conservatoire des Arts et Métiers and at the École Centrale des Arts et Manufactures in Paris.

The method used to present the elements of perspective is that of Cousinery, and is especially emphasized by Bricard. He justly maintains that this method, different from the traditional and later Monge constructions, has, in France, not received the recognition which it deserves.

The method consists in fixing the center of projection C in space by a fixed circle K in the picture plane, so that the end of the perpendicular erected at the center of K, and equal to the radius of K, coincides with C. In this manner a line l in space is determined, when its perspective l', the trace S of l on l' and the vanishing point F' of l on l', are known. The perspective P' of a point P is determined by the intersection of the perspectives l' and g' of two lines l and g through P. By this method also the homology between a figure in a plane Σ after rotating it around its trace down into the picture plane Σ', and its perspective in Σ', is easily obtained.

It may be of interest to state a few historic facts concerning this method of perspective representation. Cousinery published his *Géométrie Perspective ou Principes de Projection Polaire Appliquée à la Description des Corps* in 1828.

Brook Taylor, long before, in 1715 and 1719, made use of the same idea in *New Principles of Linear Perspective.* Joseph Highmoore applied Taylor's method in *The Practise of Perspective on the Principles of Dr. Brook Taylor*, which appeared in 1763.

Also J. H. Lambert, in 1759 and 1774, published a treatise based upon the "distance-circle:" *Die freie Perspective oder Anweisung jeden perspectivischen Aufriss von freien Stücken und ohne Grundriss zu verfertigen.*

Fiedler made systematic use of the "Distanzkreis" in his voluminous *Die Methoden der darstellenden und die Elemente der projectivischen Geometrie*, and his "*Cyclographie*," after having given an outline of his methods in the Programmabhandlung of the Gewerbeschule of Chemnitz in 1860: *Die Centralprojection als geometrische Wissenschaft.*

Bricard does not presuppose any knowledge of projective geometry and develops as much of the elements, Desargues' theorem, etc., as are needed for the understanding of what follows. The "*Petit Traité*" is an excellent introduction to the essentials of linear perspective, and it is obvious that it has been written by a competent hand.

ARNOLD EMCH

Théorie des Nombres. Tome second: Le Second Degrée Binaire, by E. Cahen, Paris, Hermann, 1924. 736 pp.

This volume is devoted to the development of that part of the theory of numbers which centers around the study of the binary quadratic forms. As the number of pages suggests, the treatment is very comprehensive and besides giving a complete account of all theory necessary in the development, problems in Diophantine analysis, which spring out of the theory of quadratic forms, are taken up for consideration when their interest is of sufficient generality to warrant it. To each chapter is appended a list of exercises in which problems of less general interest are left to the reader.

For any arithmetic theory relative to the field of rational numbers a comprehension of the classical theory of numbers is essential. The author devotes the first four chapters to the development of this theory concluding with the consideration of congruences of the second degree in one unknown. This development is followed by five chapters in which treat the idea of a number in general and various methods of approximation. Special attention is given to the theory of continued fractions and the development of numbers into continued fractions, and this again with special reference to quadratic irrationalities.

Chapter ten is a discussion of Diophantine equations in which one member is a quadratic polynomial in x and y. A very complete study is made of Fermat's equation of the first and second species

$$x^2 - Dy^2 = \pm 1, \qquad x^2 + xy - ky^2 = \pm 1.$$

The first of these is what is commonly called Pell's equation, an error of long standing.

The discussion of quadratic Diophantine equations in two unknowns is followed by a general study of binary quadratic forms in which both the algebraic and arithmetic theories are considered. The study includes the theory of the classification of binary quadratic forms and the separation of the classes into genera. The group theory with special reference to finite abelian groups is developed sufficiently for a good comprehension of the theory of composition of forms and the group of classes.

In the study of quadratic forms the author considers the general form $ax^2 + bxy + cy^2$ in place of the form $ax^2 + 2bxy + cy^2$.

Chapter 25 is a study of the equation $x^p + y^p = z^p$ for $p = 2, 3, 4$. The study of these equations leads, as is well known, to quadratic irrationalities $i = \sqrt{-1}$ and $j = (-1 + \sqrt{-3})/2$. The two fields $C(i)$ and $C(j)$ are studied in detail in chapters 26 and 27.

The remaining chapters contain the development of the general theory of quadratic number fields with the laws governing the factorization of rational primes in such fields and the classification of ideals.

G. E. Wahlin

Bibliographie de la Relativité, suivie d'un appendice sur les déterminants à deux dimensions, le calcul des variations, les séries trigonométriques, et l'azéotropisme. By Maurice Lecat. Bruxelles, Lamertin, 1924. 292+47 pp.

M. Lecat's bibliography of relativity should find a place in all university libraries and also in the personal libraries of those who are working in this field. In carefully checking the list against my own (limited) store of bibliographical information I found only one omission, viz. the pamphlet by A. N. Whitehead entitled *The Principles of Natural Science*, a lecture delivered at Bryn Mawr on the eighteenth of April, 1922, and published by Bryn Mawr College. In view of the limited circulation which this pamphlet received I feel justified in saying that M. Lecat's book is a remarkably complete index of the world's literature on relativity as it was on the day of publication, April 10, 1924.

C. N. Reynolds, Jr.

Höhere Mathematik. Teil I: *Differentialrechnung und Grundformeln der Integralrechnung nebst Anwendungen.* By R. Rothe. Leipzig, B. G. Teubner, 1925. vii+185 pp.

This little volume is one of the collection of Teubner's Technische Leitfäden, whose purpose is to give to technical students the theoretical foundations of their work in a brief and handy form.

The volume under discussion, the first of three on "Höhere Mathematik", is devoted mainly to the differential calculus. The chapter headings are: (1) Numbers, variables, and functions, (2) Main theorems of the differential calculus and fundamental integration formulas (3) functions of two or more variables, (4) differential geometry of plane curves, (5) complex numbers, variables and functions. The treatment is brief and concise, but there is no attempt to sacrifice rigor and clearness to brevity. The number of exercises is rather small, especially in comparison with similar books in this country, but on the whole they are well chosen. Of applications (and we assume this to mean applications of a technical nature) there are fewer than one would expect from the title of the book. The chief ones are references to questions of approximations, brief mention of linkages (in connection with curves in polar coordinates) and occasionally a problem in the exercises with an engineering setting.

The book is interesting as indicating perhaps what the mathematica equipment in the direction of the calculus, of an engineer in Germany is supposed to be. The opening chapter, which is really an introduction to the theory of functions of a real variable, would to seem to indicate that a student of calculus is supposed to cope successfully with a rigorous treatment of limits and continuity. American textbooks

especially those intended for engineering students, seem to avoid mentioning such concepts rigorously. The closing chapter of the book, which gives the elements of the theory of functions of a complex variable, is another indication of this broadness of scope. Perhaps American engineers would not be so overawed by differentiation and integration processes if the mathematical foundations were broader.

T. H. HILDEBRANDT

Les Erreurs Philosophiques de M. Einstein. Étude directe de la Relativité. By G. Joly. Paris, Edition Spes, 1925. 64 pp.

Amidst the mass of temporary literature evoked by the contact between the enthusiastic disciples of Einstein or of his competitors, and the perturbed conservatives of the older schools, this little book finds a natural place. It is an attack vigorously directed against the body of Einstein's conclusions.

The author cordially concedes that all of Einstein's formulas and equations are to be accepted as "guaranteed by several of the most eminent specialists," and a large part of this small book is devoted to the effort of establishing (for the restricted theory) these very formulas upon the basis of classical mechanics. A distinction is drawn between the mathematical calculations and the philosophical conclusions and the latter are rejected *in toto* by this author. The author admits what he calls the *fact* of relativity, which he claims consists in the circumstance that linear measurements reveal nothing concerning actual locations, but only relative distances. In fact he asserts incidentally that "*no point is at rest in the universe*" (italics quoted). However the author insists that the establishment of one of the formulas which he accepts as do the relativists "supposes *implicitly* comparison with a system fixed in absolute space, where lengths and times have an absolute measure". "Relativity has no sense except by comparison with the absolute, it presupposes the latter and may be deduced from it. In denying this reality, M. Einstein is led to a conception which 'is not expressible in words but only in mathematical formulas.'" "The perturbations rightly signalized by M. Einstein are most of the time so slight as to elude experimental verification. ... There is no need of a special theory to correct aberrations due to these relative movements. ... It suffices to take account of their causes in laying down the equations."

The modern student of relativity may find interest in some of the simple methods used here in discussing the elementary features of the restricted theory, but the book seems chiefly designed to aid and comfort those who cannot dispute mathematics with Einstein but who would like to feel that the old Newtonian system is as good as ever.

A. A. BENNETT

On the Direct Numerical Calculation of Elliptic Functions and Integrals. By Louis V. King. Cambridge, University Press, 1924. viii + 42 pp.

The author undertakes in this little brochure to connect the theory of elliptic functions and integrals with the so-called "arithmetico-geometric mean scales" in order to relate more directly that theory with the computing machine. Using the series of numbers (a_n, b_n) where $a_n = \frac{1}{2}(a_{n-1} + b_{n-1})$, $b_n = \sqrt{a_{n-1} b_{n-1}}$, $a_0 > b_0 > 0$ he derives computation formulas for the various elliptic functions in terms of these.

The undertaking appears to be in line with other efforts now being made to use the computing machine to avoid tedious and uncertain interpolations. Large tables will of course always be necessary and for most purposes sufficient, but the use of anything but first differences will always involve danger of errors. This little book contains in convenient form many well known formulas and also many new ones.

D. N. LEHMER

Über Paraboloide, Hyperboloide und Ellipsoide von Archimedes. (Ostwald's *Klassiker der exakten Wissenschaften*, No. 210.) Translated and annotated by A. Czwalina. Leipzig, Akademische Verlagsgesellschaft, 1923. 73 pp.

This little book consists of 66 pages of translation of Archimedes' work on quadric surfaces of revolution, together with 7 pages of notes. It will doubtless prove to be a useful addition to the materials for the study of the history of mathematics in German schools; but for English-speaking students it is without significance, on account of the existence of Sir Thomas Heath's definitive edition of Archimedes in English. The reviewer will merely state that the translation has been carefully done, and that the notes give in modern notation concise explanations of the less evident parts of Archimedes's discussion. The statement of the last theorem on page 64 contains a misprint in that the words "Hälfte der" should be inserted before "Verbindungsstrecke." Some of the figures lack portions of their lettering. Otherwise, no typographical error of consequence was noted.

R. B. MCCLENON

An Introduction to the Theory of Statistics. 7th edition. By G. Udny Yule. London, Charles Griffin and Co., Limited, 1924. xv + 415 pp.

The seventh edition of this reliable text differs very little from the preceding edition. Chapter III, on Association, has been practically rewritten; the supplementary list of references has been brought up to date; and some minor alterations and corrections have been made. Additional matter incorporated in all the recent editions has been added in the form of Supplements.

C. H. FORSYTH

NOTES

Beginning with the next issue, this BULLETIN will again be printed in the United States, at the Banta Publishing Company, Menasha, Wisconsin.

The second number of volume 27 of the TRANSACTIONS OF THIS SOCIETY (April, 1925) contains the following papers: *The subgroup composed of the substitutions which omit a letter of a transitive group*, by G. A. Miller; *On the closeness of approach of complex rational fractions to a complex irrational number*, by L. R. Ford; *Solutions of the Einstein equations involving functions of only one variable*, by E. Kasner; *A general theory of linear sets*, by M. H. Ingraham; *On the representation of a certain fundamental law of probability*, by H. L. Rietz; *The group of motions of an Einstein space*, by J. Eiesland; *A generalization of Levi-Civita's parallelism and the Frenet formulas*, by J. H. Taylor. The third number (July, 1925) contains: *The linear complex of conics*, by E. E. Libman; *On the Weddle surface and analogous loci*, by A. Emch; *Algebraic surfaces with reducible bitangent and osculating hyperplanar sections*, by M. Castellani; *Polynomials of several variables and their residue systems*, by A. J. Kempner; *A criterion for the conformal equivalence of a Riemann space to a euclidean space*, by J. Douglas; *The deflection of a rectangular plate fixed at the edges*, by H. W. March; *On irredundant sets of postutates*, by A. Church; *Three-dimensional manifolds of states of motion*, by H. Hotelling; *Relations between the critical points of a real function of n independent variables*, by M. Morse.

The joint September–December, 1924, number of the ANNALS OF MATHEMATICS ((2), vol. 26, Nos. 1–2), contains: *Contact transformations linear in x, y, z; applications to equilong transformations*, by B. H. Brown; *On the trigonometric representation of an ill-defined function*, by D. Jackson; *An irregular boundary value and expansion problem*, by L. E. Ward; *Maximal cuspidal curves*, by T. R. Hollcroft; *Plane cubics associated with the quadrangle-quadrilateral configuration*, by B. M. Turner; *On Pellet's theorem concerning the roots of a polynomial*, by J. L. Walsh; *The setting of a proposition*, by P. J. Daniell; *Theta functions and arithmetic*, by E. T. Bell; *A new type of criteria for the first case of Fermat's last theorem*, by H. S. Vandiver; *Algebraic functions and their divisors*, by G. A. Bliss; *An extension of the definition of the Green's function in one dimension*, by W. M. Whyburn; *Real representations of analytic complex curves*, by W. C. Graustein; *Note on Dirichlet series with complex exponents*, by J. F. Ritt; *On the order of an analytic function at a singular point*, by M. H. Stone;

Representations of integers in certain binary, ternary, quaternary and quinary quadratic forms and allied class number relations, by E. T. Bell. The March, 1925, number ((2), vol. 26, No. 3) contains: *Non-monoidal involutions which contain a web of invariant monoids*, by V. Snyder; *Definite integrals containing a parameter*, by R. L. Jeffery; *Integro-differential expressions invariant under Volterra's group of transformations*, by A. D. Michal; *New proofs of two well known theorems on quadratic forms*, by J. F. Ritt; *Some relations between compound determinants*, by W. H. Metzler; *A contribution to the theory of interpolation*, by N. Wiener; *A property of cyclotomic integers and its relation to Fermat's last theorem (second paper)*, by H. S. Vandiver; *On an infinite system of non-abelian groups of order* nm^n, by W. E. Edington; *Conformal and geodesic mapping*, by A. Bramley.

The second number of volume 47 of the AMERICAN JOURNAL OF MATHEMATICS (April, 1925) contains: *Some properties of the exponential mean*, by J. Dale; *The eliminant of a net of curves*, by F. Morley; *On an equation of planar motion*, by F. Morley; *An extension of the problem of the elastic bar*, by H. T. Davis; *On the Sylow subgroups of the symmetric and alternating groups*, by L. Weisner; *On certain theorems regarding summable series and their application to the double and triple Fourier's series*, by G. M. Merriman; *On the power characters of units in a cyclotomic field*, by H. S. Vandiver.

The Sociedad Matemática Argentina has begun publication of a journal entitled REVISTA MATEMÁTICA. The first number appeared in October, 1924.

Several numbers have appeared of the JAPANESE JOURNAL OF MATHEMATICS, published under the auspices of the National Research Council of Japan.

The Academy of Mathematical and Physical Sciences of the Royal Society of Naples has been commissioned by the faculty of sciences of the University of Naples to take charge of the award of the quinquennial Gabriele and Ruggero Torelli prize, for the encouragement of research in pure mathematics. Those receiving the doctorate in pure mathematics at an Italian university during the interval from January 1, 1925, to December 1, 1929, are eligible for the next award. Applicants should send copies of their dissertations and other published works to the Academy before January 2, 1930.

The Royal Society of Edinburgh has awarded the Gunning Victoria Jubilee Prize for the period 1920–24 to Sir Joseph Thomson, in recognition of his discoveries in physical science, and the MacDougall-Brisbane Prize for the same period to Professor H. Stanley Allen, for his papers on the quantum and atomic theory.

On the occasion of the celebration of the fiftieth anniversary of his entrance into teaching (see this BULLETIN, vol. 31, p. 185) Professor A. V. Vasilief was elected an honorary member of the Kharkof Mathematical Society.

Professor T. von Kármán, of the Technical School at Aachen, has been elected a corresponding member of the Göttingen Gesellschaft der Wissenschaften.

Professor A. Kneser, of the University of Breslau, has been elected a corresponding member of the Russian Academy of Sciences.

Professors Paul Koebe, of the University of Jena, and Wilhelm Wirtinger, of the University of Vienna, have been elected corresponding members of the physico-mathematical class of the Prussian Academy of Sciences.

The newly founded University of Kovno, Lithuania, conferred an honorary doctorate of mathematics on Professor Aurel Voss, of the University of Munich, on the occasion of his eightieth birthday. Professor Voss had presented his large library to the mathematical seminar of the university.

The faculty of sciences of the University of Padua is collecting a fund for a memorial to Professor Giuseppe Veronese, on the occasion of the tenth anniversary of his death.

The University of Glasgow has conferred an honorary degree on the mathematical physicist Paul Painlevé, premier of France.

Oxford University has conferred an honorary doctorate of science on Professor Niels Bohr, of the University of Copenhagen.

The University of Manchester has conferred the degree of doctor of science on Professor A. S. Eddington, of Cambridge.

Professors Niels Bohr and A. S. Eddington have been elected foreign members of the National Academy of Sciences. Professor S. Lefschetz, of Princeton University, has been elected a member in the section of mathematics.

Professor G. G. Chambers, of the University of Pennsylvania, has received the honorary degree of doctor of science from Dickinson College.

The Franklin Institute has conferred Franklin medals and certificates of honorary membership on Dr. P. Zeeman, professor of physics at the University of Amsterdam, and Dr. Elihu Thomson, of the General Electric Company.

John R. Freeman, consulting engineer, of Providence, has made a gift of securities valued at $25,000 to the Boston Society of Civil Engineers for the establishment of a fund, the income of which is to be used for encouraging research by the younger engineers of the

Society through the award of prizes for papers on hydraulics and allied subjects.

Professor E. V. Huntington, of Harvard University, is acting as Western Exchange Professor during the first half of this academic year; and will be at Beloit College, Carleton College and Know College.

Columbia University has conferred the honorary degree of doctor of science on Dr. Irving Langmuir, of the General Electric Company.

Yale University has conferred the honorary degree of doctor of science on Professor R. A. Millikan.

The Case School of Applied Science and Union College have conferred honorary degrees on Professor M. I. Pupin, of Columbia University.

Assistant Professor J. R. Kline, of the University of Pennsylvania, has been awarded a Guggenheim Fellowship, to study the analysis situs of three dimensions from a point-set standpoint, principally at the University of Göttingen.

Assistant Professor J. L. Walsh, of Harvard University, has been awarded a National Research Fellowship.

The two Benjamin Peirce Instructorships in Mathematics at Harvard University (see this Bulletin, vol. 21, p. 315) are again open for general competition. Applications for the year 1925-26, accompanied by the necessary papers, should reach Cambridge not later than February 15, 1926.

Mr. Bertrand Russell has been appointed Tarner lecturer in the philosophy of the sciences for 1925-26 at Trinity College, Cambridge.

Mr. E. F. Baxter, of the University of Sheffield, has been appointed lecturer in applied mathematics at the University of Durham.

Mr. W. L. Ferrar, of the University of Edinburgh, has been elected a fellow of Hertford College, Oxford.

Professor J. L. Synge, formerly of the University of Toronto, has been elected a fellow of Trinity College and professor of natural philosophy at the University of Dublin.

Professor C. N. Armstrong, of Ohio Wesleyan University, has been appointed head of the department of mathematics to fill the vacancy caused by the death of Professor C. B. Austin.

At West Virginia University, Assistant Professors Margaret Buchanan, C. N. Reynolds, and B. M. Turner have been promoted to associate professorships of mathematics.

Dr. Julia Dale, recently at the University of Oklahoma, has been appointed professor of mathematics at the Delta State College of Mississippi.

Assistant Professor J. E. Davis, of the Drexel Institute, has been appointed associate professor of engineering extension at Pennsylvania State College.

Dr. Philip Franklin has been promoted to an assistant professorship of mathematics at the Massachusetts Institute of Technology.

At the University of Wyoming, Professor H. C. Gossard has been appointed to a full professorship of mathematics, and Mr. O. H. Rechard has been promoted to an associate professorship.

Dr. C. C. Grove has been appointed assistant professor of mathematics at the Brooklyn Polytechnic Institute.

At the University of Minnesota, Associate Professor W. L. Hart has been promoted to a full professorship of mathematics, and Dr. Gladys Gibbens to an assistant professorship.

Assistant Professor Olive C. Hazlett, of Mount Holyoke College, has been appointed assistant professor of mathematics at the University of Illinois.

Assistant Professor A. J. Kempner, of the University of Illinois, has been appointed full professor of mathematics at the University of Colorado.

Professor Max Mason, of the University of Wisconsin, has been elected president of the University of Chicago.

Professor A. A. Michelson, of the University of Chicago, has been appointed to the first of the distinguished service professorships recently established at that university.

Assistant Professor E. J. Miles, of Yale University, has been promoted to an associate professorship of mathematics.

Mr. Max Morris has been promoted to an assistant professorship of mathematics at the Case School of Applied Science.

The marriage is announced of Professor Anna J. Pell, of Bryn Mawr College and Professor A. L. Wheeler, professor of Latin at Princeton University. They will continue to occupy their present professorships, and they will be at home after this fall in Princeton and in Bryn Mawr.

Dr. J. A. Shohat has been appointed to an assistant professorship at the University of Michigan.

Assistant Professor L. L. Smail, of the University of Oregon, has been appointed to an assistant professorship at the University of Texas.

Mr. Emory P. Starke has been promoted to an assistant professorship of mathematics at Rutgers University.

At the Texas State College at Lubbock, Dr. L. D. Ames, formerly of the University of Missouri, has been appointed associate professor of mathematics, and Miss Elizabeth Stafford has been appointed adjunct professor of mathematics.

Dr. J. H. Taylor has been appointed assistant professor of mathematics at Lehigh University.

Associate Professor H. S. Uhler, of Yale University, has been appointed head of the department of physics at Gettysburg College.

Professor H. A. Wilson, of the University of Glasgow, has accepted reappointment to the professorship of physics at Rice Institute which he held from 1912 to 1924.

Associate Professor R. M. Winger, of the University of Washington, has been promoted to a full professorship.

Professor Frederick Wood, of Lake Forest College, has been appointed professor of mathematics at Wesleyan College, Macon, Ga.

The following appointments to instructorships are announced:
American College for Women, Constantinople, Miss E. Marie Plapp;
Connecticut College for Women, Miss Mildred Carlen;
University of Florida, Mrs. Claire Harkins;
West Virginia University, Mr. H. A. Davis;
University of Wyoming, Miss Greta Neubauer.

The distinguished mathematician Felix Klein, professor of mathematics at the University of Göttingen and editor of the MATHEMATISCHE ANNALEN, died June 22, 1925, at the age of seventy-six.

Dr. Heinrich Müller-Breslau, professor of statics and building construction at the Charlottenburg Technical School, died April 23, 1925, at the age of seventy-three.

Professor Carl Neumann, of Leipzig, died March 27, 1925, in his ninety-third year.

Camille Flammarion, astronomer and author of numerous popular astronomical works, died June 4, 1925, at the age of eighty-three.

Professor Gualtiero Romboli, of the Technical School at Forli, died March 20, 1925, at the age of forty-five.

Dr. W. S. Dennett, of New York City, died March 6, 1925. Dr. Dennett had been a member of the Society since 1892, and served as its treasurer from 1900 to 1907.

Professor S. J. Lockner, of the University of Pittsburgh, died May 10, 1925, at the age of fifty-five.

Dr. Mansfield Merriman, professor of civil engineering at Lehigh University from 1878 to 1907, died June 7, 1925, at the age of seventy-seven. Dr. Merriman had been a member of the American Mathematical Society since 1891.

Professor W. A. Hamilton, of Antioch College, died June 25th, 1925.

Dr. C. E. Stromquist, for many years head of the department of mathematics at the University of Wyoming, died early in April, 1925.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

APPELL (P.). Sur une forme générale des équations de la dynamique. (Mémorial des Sciences Mathématiques, No. 1.) Paris, Gauthier-Villars, 1925. 50 pp.

BRAGG (W.). Concerning the nature of things. Six lectures delivered at the Royal Institution. London, Bell, 1925. 11+232 pp.

BRUNSCHVICG (L.). Le génie de Pascal. Paris, Hachette, 1924. 14+200 pp.

CARR (H. W.). The scientific approach to philosophy. London, Macmillan, 1924. 8+278 pp.

CRENSHAW (B. H.) and KILLEBREW (C. D.). Analytic geometry and calculus. Philadelphia, Blakiston, 1925. 10+222 pp.

FLADT (K.). Unendliche Reihen. (Mathematisch-Physikalische Bibliothek, Nr. 61.) Leipzig, Teubner, 1925. 52 pp.

GUT (M.). Ueber die singulären Moduln der Ikosaeder-Modulfunktion. (Diss., Zürich.) Zürich, 1924. 102 pp.

HADAMARD (J.). Cours d'analyse professé à l'Ecole Polytechnique. Tome 1, fascicule 1. Paris, Hermann, 1925. 336 pp.

HILBERT (D.). Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen. 2te Auflage. Leipzig, Teubner, 1924.

JAMES (R.). See LEWENT (L.).

KILLEBREW (C. D.). See CRENSHAW (B. H.).

KLOOSTERMAN (H. D.). Over het splitsen van geheele positieve getallen in een som van kwadraten. (Diss., Leiden.) Groningen, Noordhoff, 1924. 79 pp.

LEVI-CIVITA (T.). Lezioni di calcolo differenziale assoluto. Roma, Stock, 1925. 315 pp.

LEWENT (L.). Conformal representation. Translated by R. James and D. H. Williams. London, Methuen, 1925. 8+146 pp.

LOCKE (L.). The ancient quipu or Peruvian knot record. New York, Museum of Natural History, 1923. 84 pp.+59 plates.

LORIA (G.). Pagine di storia della scienza. Torino, Paravia, 1925. 15+151 pp.

MATELL (M.). Asymptotische Eigenschaften gewisser linearer Differentialgleichungen. (Diss., Uppsala.) Uppsala, 1924. 67 pp.

NICOD (J.). La géométrie dans le monde sensible. Préface de M. B. Russell. Paris, Alcan, 1924. 16+174 pp.

PÓLYA (G.) und SZEGÖ (G.). Aufgaben und Lehrsätze aus der Analysis. Band 2. (Die Grundlehren der mathematischen Wissenschaften, Band 20.) Berlin, Springer, 1925. 10+407 pp.

ROTHE (R.). Höhere Mathematik. Teil 1: Differentialrechnung und Grundformeln der Integralrechnung nebst Anwendungen. Leipzig, Teubner, 1925. 8+186 pp.

RUSSELL (B.). See NICOD (J.).

SMAIL (L. L.). History and synopsis of the theory of summable infinite processes. Eugene, University of Oregon Press, 1925. 6+175 pp.

SMITH (D. E.). History of mathematics. Volume 2: Special topics of elementary mathematics. Boston, Ginn, 1925.

SZEGŐ (G.). See PÓLYA (G.).

UNGERER (E.). Wissen und Wirken. Karlsruhe, O. Braun, 1924.

VERRIEST (G.). Cours de mathématiques générales. Partie 2. Paris, Gauthier-Villars, 1925. 388 pp.

WILLIAMS (D. H.). See LEWENT (L.).

PART II. APPLIED MATHEMATICS

ACKERET (J.). Das Rotorschiff und seine physikalischen Grundlagen. Göttingen, Vandenhoeck und Rupprecht, 1925. 48 pp.

BELLASIS (E. S.). Hydraulics with working tables. 4th edition, revised. London, E. and F. N. Spon, 1924. 356 pp.

— River and canal engineering. 2d edition, revised. London, E. and F. N. Spon, 1924.

BERKELEY (L. M.). North star navigation. New York, White Book and Supply Company, 1924. 86 pp.

BOHR (N.). The theory of spectra and atomic constitution. Three essays. 2d edition. Cambridge, University Press, 1924.

TYCHONIS BRAHE Dani opera omnia. Edidit I. L. E. Dreyer. Tomus 11. Hauniae, Libraria Gyldendaliana, 1924. 5+422 pp.

CAMPBELL (N. R.). La structure de l'atome. 2e supplément à l'ouvrage La théorie électrique moderne. Traduit de l'anglais par A. Corvisy. Paris, Hermann, 1925. 6+166 pp.

CHAUVEAU (B.). Electricité atmosphérique. Fascicule 2: Le champ électrique de l'atmosphère. Paris, Doin, 1925. 10+264 pp.

CLARKE (J. R.). See GUYE (C. E.).

CORPS (—.). Le camouflage de la simultanéité, base unique des théories de la relativité. Pontoise, Imprimerie Lucien Paris, 1924. 12 pp.

CORVISY (A.). See CAMPBELL (N. R.).

DE COUSSANGE (J.). See HOFFDING (H.).

CURRIER (E. L.), LENNES (N. J.) and MERRILL (A. S.). Farm accounting. New York, Macmillan, 1924. 9+287 pp.

DARRAS (M.). Statique graphique élémentaire et notions préliminaires de résistance des matériaux. Paris, Librairie Centrale des Sciences, 1925. 397 pp.

DE DONDER (T.). La gravifique de Weyl-Eddington-Einstein. Paris, Gauthier-Villars, 1924. 32 + 16 pp.

Théorie mathématique de l'électricité. Partie 1: Introduction aux équations de Maxwell. Paris, Gauthier-Villars, 1925. 200 pp.

DREYER (I. L. E.). See TYCHO BRAHE.

DUPONT (P.). La mécanique nouvelle démontrée par les principes classiques. Interpretation et transformation des équations de Lorentz et d'Einstein. Paris, Hermann, 1925. 14 + 150 pp.

EXNER (F. M.). Dynamische Meteorologie. Berlin, Springer, 1925. 8 + 421 pp.

FABER (O.). Reinforced concrete beams in bending and shear. Theory and tests in support. London, Concrete Publications, Ltd., 1924. 11 + 150 pp.

FOURNIER D'ALBE (E. E.). See NORDMANN (C.).

GEORGII (W.). Wettervorhersage. Dresden, Steinkopff, 1924. 114 pp.

GOULDEN (C.). Refraction of the eye, including elementary physiological optics. London, Churchill, 1925. 12 + 276 pp.

GREINACHER (H.). Ionen und Elektronen. Leipzig, Teubner, 1924. 58 pp.

GUYE (C. E.). Physico-chemical evolution. Translated by J. R. Clarke. London, Methuen, 1925. 12 + 172 pp.

HAHN (K.). Grundriss der Physik. 2te Auflage. Teil 2. Leipzig. Teubner, 1924.

HARVEY (F. W.). Examples in mechanics. London, Methuen, 1925. 6 + 82 pp.

HAZELTINE (L. A.). Electrical engineering. New York, Macmillan, 1924. 16 + 625 pp.

HEHRE (F. W.). See MORECROFT (J. H.).

HENRY (C.). Essai de généralisation de la théorie du rayonnement. Résonateurs gravifiques et résonateurs biologiques. Exposition intuitive des résultats techniques essentiels de la théorie de la relativité. Paris, Hermann, 1925. 144 pp.

HOBSON (E. W.). The ideal aim of physical science. Cambridge, University Press, 1925. 4 + 34 pp.

HOFFDING (H.). La relativité philosophique. Traduit du danois par J. de Coussange. Paris, Alcan, 1924.

HORT (W.). Die Differentialgleichungen des Ingenieurs. 2te umgearbeitete und vermehrte Auflage. Berlin, Springer, 1925. 12 + 700 pp.

JOLLEY (L. B. W.). Alternate current rectification. A mathematical and practical treatment from the engineering view-point. London, Chapman and Hall, 1924. 18 + 352 pp.

JONES (F. W.). The Hodsock ballistic tables for rifles. London, Arnold, 1925. 8 + 82 pp.

KENT (F. C.). Elements of statistics. New York, McGraw-Hill, 1924. 11 + 178 pp.

KOHLHÖRSTER (W.). Die durchdringende Strahlung der Atmosphäre. Hamburg, H. Grand, 1924. 72 pp.

LAPORTE (M.). Le radium. Paris, Stock, 1924. 128 pp.

LENNES (N. J.). See CURRIER (E. L.).

LENZ (K.). Die Rechenmaschinen und das Maschinenrechnen. 2te Auflage. Leipzig, Teubner, 1924. 6 + 108 pp.

MCLAREN (S. B.). Scientific papers mainly on electrodynamics and natural radiation including the substance of an Adams prize essay in the University of Cambridge. Cambridge, University Press, 1925. 8 + 112 pp.

MENGES (C. L. R. E.). Nouvelles vues Faraday-Maxwelliennes. Supplément: Sur la propagation de la lumière. Paris, Gauthier-Villars, 1924. 6 + 44 pp.

MERRILL (A. S.). See CURRIER (E. L.).

MEYERSON (E.). La déduction relativiste. Paris, Payot, 1925. 396 pp.

MORECROFT (J. H.) and HEHRE (F. W.). Electrical circuits and machinery. Volume 2: Alternating currents. New York, Wiley, 1924. 11 + 444 pp.

NIEWENGLOWSKI (G. H.). Les rayons X et le radium. Paris, Hachette, 1925. 184 pp.

NORDMANN (C.). The tyranny of time. Einstein or Bergson? Translated from the French by E. E. Fournier d'Albe. London, T. Fisher Unwin, 1925. 215 pp.

PRZIBRAM (R.). Aufbau mathematischer Biologie. Berlin, Gebrüder Borntraeger, 1923.

ROUCH (J.). Les méthodes de prévision du temps. Paris, Alcan, 1924. 280 pp.

SALLINGER (F.). Die Gleichstrommaschine. 1ter und 2ter Teil. (Sammlung Göschen.) Berlin, de Gruyter, 1923–24.

SCHWENGLER (J.). Der Bau der Starrluftschiffe. Ein Leitfaden für Konstrukteure und Statiker. Berlin, Springer, 1925. 103 pp.

SHEARCROFT (W. F. F.). The story of the atom. London, Benn, 1925. 79 pp.

DE SPARRE (-.). Calcul du temps et de la dérivation dans les grandes trajectoires des projectiles. Paris, Gauthier-Villars, 1924.

TAYLOR (H. S.). A treatise on physical chemistry. Volumes 1–2. London, Macmillan, 1924. 655 + 709 pp.

THIRTY-FOURTH ANNUAL LIST OF PAPERS

READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES TO THE PLACES OF PUBLICATION

ALEXANDER, J. W. Topological invariants of manifolds. Read Oct. 25, 1924. *Proceedings of the National Academy of Sciences*, vol. 10, No. 12, pp. 493–494; Dec., 1924.

BARNETT, I. A. On a class of transformations in function space. Read April 14, 1923. *Annals of Mathematics*, (2), vol. 25, No. 3, pp. 205–220; March, 1924.

—— Integro-differential equations. Read Dec. 29, 1923. *This Bulletin*, vol. 31, Nos. 5–6, pp. 263–265; May–June, 1925.

BELL, E. T. Certain products involving the divisors of numbers. Read (San Francisco Section) April 5, 1924. *Bulletin of the Calcutta Mathematical Society*, vol. 15, No. 1, pp. 21–24; June, 1924.

—— Theta functions and arithmetic. Read (San Francisco Section) April 7, 1923. *Annals of Mathematics*, (2), vol. 26, Nos. 1–2, pp. 79–87; Sept.–Dec., 1924.

—— Representations of integers in certain binary, ternary, quaternary and quinary quadratic forms and allied class number relations. Read Dec. 27, 1923. *Annals of Mathematics*, (2), vol. 26, Nos. 1–2, pp. 155–164; Sept.–Dec., 1924.

—— Notes on recurring series of the third order. Read (San Francisco Section) Sept. 18, 1923. *Tôhoku Mathematical Journal*, vol. 24, Nos. 1–2, pp. 168–184; Dec., 1924.

—— On the number of representations of an integer as a sum or difference of two cubes. Read (San Francisco Section) June 19, 1925. *This Bulletin*, vol. 31, No. 7, pp. 309–312; July, 1925.

BENNETT, A. A. Five axioms for point and translation in affine geometry. Read Dec. 30, 1924. *This Bulletin*, vol. 30, Nos. 9–10, pp. 520–526; Nov.–Dec., 1924.

—— On sets of three consecutive integers which are quadratic residues of primes. Read April 10, 1925. *This Bulletin*, vol. 31, No. 8, pp. 411–412; Oct., 1925.

BLISS, G. A. Algebraic functions and their divisors. Read April 19, 1924. *Annals of Mathematics*, (2), vol. 26, Nos. 1–2, pp. 95–124; Sept.–Dec., 1924.

BRINKMANN, H. W. Einstein spaces which are mapped conformally on each other. Read April 28, 1923. *Mathematische Annalen*, vol. 94, Nos. 1–2, pp. 119–145; April, 1925.

BROWN, B. H. Contact transformations linear in x, y, z; applications to equilong transformations. Read Dec. 27, 1922. *Annals of Mathematics*, (2), vol. 26, Nos. 1–2, pp. 1–7; Sept.–Dec., 1924.

CAJORI, F. Empirical generalizations on the growth of mathematical notations. Read (San Francisco Section) Sept. 18, 1923. *Isis*, vol. 6, No. 3, pp. 391–394; 1924.

—— Note on our sign of equality. Read (San Francisco Section) April 5, 1924. *Isis*, vol. 6, No. 4, pp. 507–508; 1924.

CAJORI, F. Notes on Luca Pacioli's "Summa." Read (San Francisco Section) April 5, 1924. *Archivio di Storia della Scienza*, vol. 5, No. 2, pp. 125–130; June, 1924.

—— Despiau's Select Amusements. Read (San Francisco Section) April 5, 1924. *Mathematical Gazette*, vol. 12, No. 172, p. 216; Oct., 1924.

—— Fanciful hypotheses on the origin of the numeral forms. Read (San Francisco Section) Oct. 25, 1924. *Mathematics Teacher*, vol. 18, No. 3, pp. 129–133; March, 1925.

—— Early printed forms of our dollar mark. Read (San Francisco Section) April 4, 1925. *School and Society*, vol. 21, No. 543, pp. 625–626; May 23, 1925.

CAMP, B. H. Probability integrals for a hypergeometric series. Read Oct. 25, 1924. *Biometrica*, vol. 17, Nos. 1–2, pp. 61–67; June, 1925.

CARIS, P. A. A solution of the quadratic congruence modulo p, $p = 8n+1$, n odd. Read April 10, 1925. *American Mathematical Monthly*, vol. 32, No. 6, pp. 294–299; June–July, 1925.

CARPENTER, A. F. Projective properties of a ruled surface in the neighborhood of a ruling. Read (San Francisco Section) Sept. 18, 1923, and (San Francisco Section) Dec. 22, 1923. *American Journal of Mathematics*, vol. 46, No. 4, pp. 241–257; Oct., 1924.

—— Cone-cubic configurations of a ruled surface. Read (San Francisco Section) April 5, 1924. *Transactions of this Society*, vol. 27, No. 4, pp. 397–415; Oct., 1925.

CASTELLANI, M. Algebraic surfaces with reducible bitangent and osculating hyperplanar sections. Read May 2, 1925. *Transactions of this Society*, vol. 27, No. 3, pp. 279–286; July, 1925.

CHURCH, A. On irredundant sets of postulates. Read Dec. 30, 1924. *Transactions of this Society*, vol. 27, No. 3, pp. 318–328; July, 1925.

CUMMINGS, L. D. A new type of double sextette closed under a binary (3,3) correspondence. Read Oct. 25, 1924. *This Bulletin*, vol. 31, Nos. 5–6, pp. 266–274; May–June, 1925.

DAVIS, H. T. An extension of the problem of the elastic bar. Read Dec. 29, 1922, and Dec. 26, 1924. *American Journal of Mathematics*, vol. 47, No. 2, pp. 101–120; April, 1925.

DICKSON, L. E. Differential equations from the group standpoint. Read April 15, 1922. *Annals of Mathematics*, (2), vol. 25, No. 4, pp. 287–378; June, 1924.

DODD, E. L. The frequency law of a function of one variable. Read Dec. 30, 1924. *This Bulletin*, vol. 31, Nos. 1–2, pp. 27–31; Jan.–Feb., 1925.

DOUGLAS, J. A criterion for the conformal equivalence of a Riemann space to a euclidean space. Read Oct. 27, 1923. *Transactions of this Society*, vol. 27, No. 3, pp. 299–306; July, 1925.

EDINGTON, W. E. On an infinite system of non-abelian groups of order nm^n. Read Dec. 29, 1922. *Annals of Mathematics*, (2), vol. 26, No. 3, pp. 233–238; March, 1925.

EIESLAND, J. The group of motions of an Einstein space. Read March 26, 1921, Sept. 8, 1921, and Sept. 7, 1923. *Transactions of this Society*, vol. 27, No. 2, pp. 213–245; April, 1925.

EISENHART, L. P. Fields of parallel vectors in a Riemannian geometry. Read Feb. 28, 1925. *Transactions of this Society*, vol. 27, No. 4, pp. 563–573; Oct., 1925.

EMCH, A. On plane algebraic curves which remain invariant under a quadratic Cremona transformation, their classification, and relation to the Cayley cubic surface. Read Dec. 28, 1921. *Tôhoku Mathematical Journal*, vol. 24, Nos. 1–2, pp. 68–87; Dec., 1924.

—— Determination of plane algebraic curves which are invariant under involutory Cremona transformations. Read April 10, 1925. *Tôhoku Mathematical Journal*, vol. 25, Nos. 1–2, pp. 63–76; May, 1925.

—— On the Weddle surface and analogous loci. Read Dec. 29, 1923. *Transactions of this Society*, vol. 27, No. 3, pp. 270–278; July, 1925.

FORD, L. R. On the closeness of approach of complex rational fractions to a complex irrational number. Read April 24, 1920. *Transactions of this Society*, vol. 27, No. 2, pp. 146–154; April, 1925.

—— The fundamental region of a Fuchsian group. Read Sept. 11, 1925. *This Bulletin*, vol. 31, Nos. 9–10, pp. 531–539; Nov.–Dec., 1925.

FRANKLIN, P. Analytic transformations of everywhere dense point sets, Read May 3, 1924. *Transactions of this Society*, vol. 27, No. 1. pp. 91–100; Jan., 1925.

—— The rotating disc. Read Oct. 25, 1924. *Proceedings of the National Academy of Sciences*, vol. 11, No. 2, pp. 147–149; Feb., 1925.

—— Functions with an essential singularity. Read May 3, 1924. *This Bulletin*, vol. 31, Nos. 3–4, pp. 157–162; March–April, 1925.

—— The electric currents in a network. Read Oct. 25, 1924. *Journal of Mathematics and Physics of the Massachusetts Institute of Technology*, vol. 4, No. 2, pp. 97–102; April, 1925.

—— The Weierstrass approximation theorem. Read Dec. 30, 1924. *Journal of Mathematics and Physics of the Massachusetts Institute of Technology*, vol. 4, No. 3, pp. 148–152; May, 1925.

GARABEDIAN, C. A. Quatre méthodes pour résoudre le problème de la poutre rectangulaire. Read Oct. 25, 1924. *Comptes Rendus de l'Académie des Sciences*, vol. 179, No. 7, pp. 381–384; Aug. 18, 1924.

—— Rods of constant or variable cross section. Read April 28, 1923. *American Journal of Mathematics*, vol. 46, No. 4, pp. 273–287; Oct., 1924.

—— Solution du problème de la plaque rectangulaire épaisse encastrée ou posée, portant une charge uniformément répartie ou concentrée en son centre. Read Feb. 28, 1925. *Comptes Rendus de l'Académie des Sciences*, vol. 180, No. 4, pp. 257–259; Jan. 26, 1925.

GLENN, O. E. A note on the abundance of differential combinants in a fundamental system. Read May 3, 1924. *Proceedings of the National Academy of Sciences*, vol. 11, No. 6, pp. 281–284; June, 1925.

GRAUSTEIN, W. C. Real representations of analytic complex curves. Read Sept. 7, 1922. *Annals of Mathematics*, (2), vol. 26, Nos. 1–2. pp. 131–143; Sept.–Dec., 1924.

GRIFFITHS, L. W. Contact curves of the rational plane cubic. Read (San Francisco Section) Dec. 22, 1923. *This Bulletin*, vol. 31, No. 7, pp. 312–317; July, 1925.

GRONWALL, T. H. The mutual inductance of two square coils. Read April 28, 1923. *Transactions of this Society*, vol. 27, No. 4, pp. 516–536; Oct., 1925.

HANCOCK, H. The foundations of the theory of algebraic numbers. Read Dec. 31, 1924. *Science*, new ser., vol. 61, No. 1566, pp. 5–10, and No. 1567, pp. 30–35; Jan. 2 and Jan. 9, 1925.

HEDRICK, E. R., and INGOLD, L. Analytic functions in three dimensions. Read Sept. 7, 1923. *Transactions of this Society*, vol. 27, No. 4, pp. 551–555; Oct., 1925.

—— The Beltrami equations in three dimensions. Read (Southwestern Section) Dec. 1, 1923. *Transactions of this Society*, vol. 27, No. 4, pp. 556–562; Oct., 1925.

HENDERSON, A. Is the universe finite? Read Jan. 1, 1925. *American Mathematical Monthly*, vol. 32, No. 5, pp. 213–223; May, 1925.

HENDERSON, R. Life insurance as a social service and as a mathematical problem. Read Dec. 30, 1924. *This Bulletin*, vol. 31, Nos. 5–6, pp. 227–252; May–June, 1925.

HILLE, E. Note on Dirichlet's series with complex exponents. Read April 28, 1923. *Annals of Mathematics*, (2), vol. 25, No. 3, pp. 261–278; March, 1924.

—— A general type of singular points. Read Oct. 25, 1924. *Proceedings of the National Academy of Sciences*, vol. 10, No. 12, pp. 488–493; Dec., 1924.

—— On the zeros of the functions of the parabolic cylinder. Read Dec. 27, 1923. *Arkiv for Matematik, Astronomi och Fysik*, vol. 18, No. 26, pp. 1–56; Jan., 1925.

—— A note on regular singular points. Read Oct. 25, 1924. *Arkiv for Matematik, Astronomi och Fysik*, vol. 19 A, No. 2, pp. 1–21; April, 1925.

HOLLCROFT, T. R. Maximal cuspidal curves. Read Feb. 25, 1922. *Annals of Mathematics*, (2), vol. 26, Nos. 1–2, pp. 37–46; Sept.–Dec., 1924.

—— Limits for actual double points of space curves. Read Feb. 24, 1923. *This Bulletin*, vol. 31, Nos. 1–2, pp. 42–55; Jan.–Feb., 1925.

HOPKINS, L. A. On periodic orbits for the characteristic planets. Read Sept. 7, 1920. *Astronomical Journal*, vol. 36, Nos. 4–5, pp. 25–33; Nov. 11, 1924.

HOTELLING, H. Three-dimensional manifolds of states of motion. Read (San Francisco Section) Oct. 25, 1924. *Transactions of this Society*, vol. 27, No. 3, pp. 329–344; July, 1925.

HUBER, C. M. On the prime divisors of the cyclotomic functions. Read Oct. 25, 1924. *Transactions of this Society*, vol. 27, No. 1, pp. 43–48; Jan., 1925.

HURWITZ, W. A. Characteristic parameter values for an integral equation. Read Dec. 30, 1924. *This Bulletin*, vol. 31, Nos. 9–10, pp. 503–508; Nov.–Dec., 1925.

HUTCHINSON, J. I. On the roots of the Riemann zeta function. Read Oct. 25, 1924. *Transactions of this Society*, vol. 27, No. 1, pp. 49–60; Jan., 1925.

INGOLD, L. Associated types of linear connection. Read April 11, 1925. *Proceedings of the National Academy of Sciences*, vol. 11, No. 5, pp. 252–256; May, 1925.

—— A symbolic treatment of the geometry of hyperspace. Read (Southwestern Section) Dec. 1, 1923. *Transactions of this Society*, vol. 27, No. 4, pp. 574–599; Oct., 1925.

—— See HEDRICK, E. R.

INGRAHAM, M. H. A general theory of linear sets. Read April 11, 1925. *Transactions of this Society*, vol. 27, No. 2, pp. 163–196; April, 1925.

JACKSON, D. On the method of least *m*th powers for a set of simultaneous equations. Read Dec. 30, 1920. *Annals of Mathematics*, (2), vol. 25, No. 3, pp. 185–192; March, 1924.

—— A general class of problems in approximation. Read Dec. 29, 1922. *American Journal of Mathematics*, vol. 46, No. 4, pp. 215–234; Oct., 1924.

—— On the trigonometric representation of an ill-defined function. Read Dec. 30, 1920. *Annals of Mathematics*, (2), vol. 26, Nos. 1–2, pp. 8–20; Sept.–Dec., 1924.

—— The elementary geometry of function space. Read April 19, 1924. *American Mathematical Monthly*, vol. 31, No. 10, pp. 461–471; Dec., 1924.

—— The geometry of frequency functions. Read Oct. 25, 1924. *This Bulletin*, vol. 31, Nos. 1–2, pp. 63–73; Jan.–Feb., 1925.

JAMES, G. An algebraically reducible solution of the cubic equation, Read Feb. 28, 1925. *American Mathematical Monthly*, vol. 32. No. 4, pp. 162–169; April, 1925.

KASNER, E. An algebraic solution of the Einstein equations. Read Oct. 29, 1921. *Transactions of this Society*, vol. 27, No. 1, pp. 101–105; Jan., 1925.

—— Separable quadratic differential forms and Einstein solutions. Read May 3, 1924. *Proceedings of the National Academy of Sciences*, vol. 11, No. 1, pp. 95–96; Jan., 1925.

—— Solutions of the Einstein equations involving functions of only one variable. Read Sept. 7, 1921. *Transactions of this Society*, vol. 27, No. 2, pp. 155–162; April, 1925.

KEMPNER, A. J. Polynomials of several variables and their residue systems. Read Dec. 26, 1924. *Transactions of this Society*, vol. 27, No. 3, pp. 287–298; July, 1925.

KEYSER, C. J. Concerning groups of dyadic relations of an arbitrary field. Read April 26, 1919. *Fundamenta Malhematicae*, vol. 7, pp. 323–339; 1925.

KLINE, J. R. Concerning the complementary intervals of countable closed sets. Read Oct. 25, 1924. *This Bulletin*, vol. 31, No. 8, pp. 409–410; Oct., 1925.

—— Concerning the sum of two continua each irreducible between the same pair of points. Read May 3, 1924. *Fundamenta Mathematicae*, vol. 7, pp. 314–322; 1925.

LEHMER, D. N. A new method of factorization. Read (San Francisco Section) April 4, 1925. *Proceedings of the National Academy of Sciences*, vol. 11, No. 1, pp. 97–98; Jan., 1925.

—— Note on the construction of tables of linear forms. Read (San Francisco Section) June 19, 1925. *This Bulletin*, vol. 31, Nos. 9–10, pp. 497–498; Nov.–Dec., 1925.

LEVY, H. Normal congruences of curves in Riemann space. Read April 28, 1923. *This Bulletin*, vol. 31, Nos. 1–2, pp. 39–42; Jan.–Feb., 1925.

—— Ricci's coefficients of rotation. Read May 3, 1924. *This Bulletin*, vol. 31, Nos. 3–4, pp. 142–145; March–April, 1925.

LIBMAN, E. E. The linear complex of conics. Read Feb. 28, 1925. *Transactions of this Society*, vol. 27, No. 3, pp. 265–269; July, 1925.

LOGSDON, M. I. Complete groups of points on a plane cubic curve. Read Dec. 29; 1922. *Transactions of this Society*, vol. 27, No. 4, pp. 474–490; Oct., 1925.

MACDUFFEE, C. C. On the complete independence of the functional equations of involution. Read April 28, 1923. *This Bulletin*, vol. 31, Nos. 1–2, pp. 21–26; Jan.–Feb., 1925.

MACMILLAN, W. D. Some mathematical aspects of cosmology. Read April 10, 1925. *Science*, new ser., vol. 62, No. 1595, pp. 63–72, No. 1596, pp. 96–99, and No. 1597, pp. 121–127; July 24, July 31, and Aug. 7, 1925.

MARCH, H. W. Deflection of a rectangular plate fixed at the edges. Read Dec. 29, 1923. *Transactions of this Society*, vol. 27, No. 3, pp. 307–317; July, 1925.

MERRIMAN, G. M. On certain theorems regarding summable series and their application to the double and triple Fourier's series. Read Dec. 28, 1923, and Dec. 29, 1924. *American Journal of Mathematics*, vol. 47, No. 2, pp. 125–139; April, 1925.

MICHAL, A. D. Functionals of curves admitting one-parameter groups of infinitesimal point transformations. Read Dec. 28, 1923. *Proceedings of the National Academy of Sciences*, vol. 11, No. 1, pp. 98–101; Jan., 1925.

—— Integro-differential expressions invariant under Volterra's group of transformations. Read April 28, 1923. *Annals of Mathematics*, (2), vol. 26, No. 3, pp. 181–201; March, 1925.

—— Functional invariants, with a continuity of order p, of one-parameter Fredholm and Volterra transformation groups. Read Oct. 25, 1924. *This Bulletin*, vol. 31, No. 7, pp. 335–346; July, 1925.

MILLER, G. A. The subgroup composed of the substitutions which omit a letter of a transitive group. Read Dec. 26, 1924. *Transactions of this Society*, vol. 27, No. 2, pp. 137–145; April, 1925.

MOORE, C. L. E. Minimal varieties of two or three dimensions whose element of arc is a perfect square. Read Feb. 28, 1925. *Journal of Mathematics and Physics of the Massachusetts Institute of Technology*, vol. 4, No. 3, pp. 167–178; May, 1925.

MOORE, C. N. On the application of Borel's method to the summation of Fourier's series. Read Dec. 29, 1924. *Proceedings of the National Academy of Sciences*, vol. 11, No. 6, pp. 284–287; June, 1925.

MOORE, R. L. Concerning upper semi-continuous collections of continua which do not separate a given continuum. Read (Southwestern Section) Nov. 29, 1924. *Proceedings of the National Academy of Sciences*, vol. 10, No. 8, pp. 356–360; Aug., 1924.

—— Concerning sets of segments which cover a point set in the Vitali sense. Read (Southwestern Section) Nov. 29, 1924. *Proceedings of the National Academy of Sciences*, vol. 10, No. 11, pp. 464–467; Nov., 1924.

—— Concerning the prime parts of a continuum. Read Sept. 7, 1923. *Mathematische Zeitschrift*, vol. 22, Nos. 3–4, pp. 307–315; Feb., 1925.

—— A characterization of a continuous curve. Read May 3, 1924. *Fundamenta Mathematicae*, vol. 7, pp. 302–307; 1925.

MOORE, R. L. Concerning the separation of point sets by curves. Read Sept. 10, 1925. *Proceedings of the National Academy of Sciences*, vol. 11, No. 8, pp. 469–476; Aug., 1925.

—— Concerning upper semi-continuous collections of continua. Read Dec. 30, 1924. *Transactions of this Society*, vol. 27, No. 4, pp. 416–428; Oct., 1925.

MORITZ, R. E. On the kinematic construction of certain higher plane curves. Read (San Francisco Section) April 4, 1925. *American Mathematical Monthly*, vol. 32, No. 6, pp. 302–305; June–July, 1925.

MORSE, M. Relations between the critical points of a real function of n independent variables. Read Dec. 28, 1923. *Transactions of this Society*, vol. 27, No. 3, pp. 345–396; July, 1925.

MURNAGHAN, F. D. The tensor character of the generalized Kronecker symbol. Read March 1, 1924. *This Bulletin*, vol. 31, No. 7, pp. 323–329; July, 1925.

MURRAY, F. H. Generalization of certain theorems of Bohl. Read March 1, 1924. *American Journal of Mathematics*, vol. 47, No. 1, pp. 25–44; Jan., 1925.

OLSON, H. L. Congruences with constant absolute invariants. Read Feb. 28, 1925. *Transactions of this Society*, vol. 27, No. 1, pp. 15–42; Jan., 1925.

OSGOOD, W. F. On normal forms of differential equations. Read Feb. 28, 1925. *Transactions of this Society*, vol. 27, No. 1, pp. 1–14; Jan., 1925.

PUPIN, M. I. From chaos to cosmos. Read Feb. 29, 1924. *Scribner's Magazine*, vol. 76, No. 1, pp. 3–10; July, 1924.

RAINICH, G. Y. Electrodynamics in the general relativity theory. Read Feb. 24, 1923, Dec. 27, 1923, March 1, 1924, May 3, 1924, and Oct. 25, 1924. *Transactions of this Society*, vol. 27, No. 1, pp. 106–136; Jan., 1925.

—— Second paper on tensor analysis. Read March 1, 1924. *American Journal of Mathematics*, vol. 47, No. 1, pp. 11–24; Jan., 1925.

—— Sur une représentation des surfaces. Read Dec. 29, 1924. *Comptes Rendus de l'Académie des Sciences*, vol. 180, No. 11, pp. 801–803; March 16, 1925.

REILLY, J. F. Certain generalizations of osculatory interpolation, Read April 14, 1922. *The Record, American Institute of Actuaries*. vol. 13, No. 1, pp. 4–22; June, 1924.

—— On Lidstone's demonstration of the osculatory interpolation formula. Read (Southwestern Section) Nov. 29, 1924. *The Record, American Institute of Actuaries*, vol. 14, No. 1, pp. 12–20; June, 1925.

REINSCH, B. P. Expansion problems in connection with the hypergeometric differential equation. Read April 19, 1924. *American Journal of Mathematics*, vol. 47, No. 1, pp. 45–70; Jan., 1925.

RIETZ, H. L. On the representation of a certain fundamental law of probability. Read April 13, 1923. *Transactions of this Society*, vol. 27, No. 2, pp. 197–212; April, 1925.

RITT, J. F. Note on Dirichlet's series with complex exponents. Read Dec. 28, 1923. *Annals of Mathematics*, (2), vol. 26, Nos. 1–2, p. 144; Sept.–Dec., 1924.

RITT, J. F. Elementary functions and their inverses. Read Oct. 25, 1924. *Transactions of this Society*, vol. 27, No. 1, pp. 68–90; Jan., 1925.
—— New proofs of two well known theorems on quadratic forms. Read March 1, 1924. *Annals of Mathematics*, (2), vol. 26, No. 3, pp. 202–204; March, 1925.

ROE, E. D. Integral functions as products. Read Sept. 5, 1917. *Publications of the Roe Observatory*, No. 1, pp. 1–6; Jan., 1924.

ROEVER, W. H. Some phases of descriptive geometry. Read (Southwestern Section) Nov. 29, 1924. *This Bulletin*, vol. 31, Nos. 9–10, pp. 540–550; Nov.–Dec., 1925.

ROMIG, H. G. Early history of division by zero. Read (San Francisco Section) Sept. 18, 1923. *American Mathematical Monthly*, vol. 31, No. 8, pp. 387–389; Oct., 1924.

SCHOUTEN, J. A. On the conditions of integrability of covariant differential equations. Read Sept. 10, 1925. *Transactions of this Society*, vol. 27, No. 4, pp. 441–473; Oct., 1925.

SHOHAT, J. A. On the development of a continuous function into a series of Tchebycheff polynomials. Read April 19, 1924. *Transactions of this Society*, vol. 27, No. 4, pp. 537–550; Oct., 1925.
—— On the polynomial of the best approximation to a given continuous function. Read Dec. 26, 1924. *This Bulletin*, vol. 31, Nos. 9–10; pp. 509–514; Nov.–Dec., 1925.

SMAIL, L. L. History and synopsis of the theory of summable infinite processes. Read (San Francisco Section) June 18, 1920, and (San Francisco Section) April 7, 1923. *University of Oregon Publication*, vol. 2, No. 8, 6+175 pp.; Feb., 1925.

SMITH, H. L. On the existence of the Stieltjes integral. Read Dec. 29, 1923. *Transactions of this Society*, vol. 27, No. 4, pp. 491–515; Oct., 1925.

SNYDER, V. On the types of monoidal involutions. Read Dec. 27, 1923. *Annals of Mathematics*, (2), vol. 25, No. 3, pp. 279–284; March, 1924.
—— Non-monoidal involutions which contain a web of invariant monoids. Read April 18, 1924. *Annals of Mathematics*, (2), vol. 26, No. 3, pp. 165-172; March, 1925.

STONE, M. H. On the order of an analytic function at a singular point. Read March 1, 1924. *Annals of Mathematics*, (2), vol. 26, Nos. 1–2, pp. 145–154; Sept.–Dec., 1924.

SYNGE, J. L. A generalization of the Riemannian line-element. Read Dec. 30, 1924. *Transactions of this Society*, vol. 27, No. 1, pp. 61–67; Jan., 1925.

TAYLOR, J. H. A generalization of Levi-Civita's parallelism and the Frenet formulas. Read April 19, 1924. *Transactions of this Society*, vol. 27, No. 2, pp. 246–264; April, 1925.
—— Reduction of Euler's equations to a canonical form. Read Oct. 25, 1924. *This Bulletin*, vol. 31, Nos. 5-6, pp. 257–262; May–June, 1925.

THOMAS, J. M. Congruences of circles studied with reference to the surface of centers. Read Sept. 7, 1923. *Annals of Mathematics*, (2), vol. 25, No. 3, pp. 221–237; March, 1924.
—— Note on the projective geometry of paths. Read Feb. 28, 1925. *Proceedings of the National Academy of Sciences*, vol. 11, No. 4, pp. 207–209; April, 1925.

THOMAS, J. M. Conformal correspondence of Riemann spaces. Read May 2, 1925. *Proceedings of the National Academy of Sciences*, vol. 11, No. 5, pp. 257–259; May, 1925.

—— The number of even and odd absolute permutations of n letters. Read Feb. 28, 1925. *This Bulletin*, vol. 31, No. 7, pp. 303–304; July, 1925.

—— See VEBLEN, O.

VANDIVER, H. S. A new type of criteria for the first case of Fermat's last theorem. Read Sept. 7, 1923. *Annals of Mathematics*, (2), vol. 26, Nos. 1–2, pp. 88–94; Sept.–Dez., 1924.

—— On sets of three consecutive integers which are quadratic or cubic residues of primes. Read Dec. 29, 1923. *This Bulletin*, vol. 31, Nos. 1–2, pp. 33–38; Jan.–Feb., 1925.

—— A property of cyclotomic integers and its relation to Fermat's last theorem. (Second paper.) Read April 24, 1915, and April 27, 1918. *Annals of Mathematics*, (2), vol. 26, No. 3, pp. 217–232; March, 1925.

—— On the power characters of units in a cyclotomic field. Read Dec. 28, 1916. *American Journal of Mathematics*, vol. 47, No. 2, pp. 140–147; April, 1925.

—— Laws of reciprocity and the first case of Fermat's last theorem. Read Sept. 11, 1925. *Proceedings of the National Academy of Sciences*, vol. 11, No. 6, pp. 292–298; June, 1925.

—— On the distribution of quadratic and higher residues. Read Oct. 28, 1916. *This Bulletin*, vol. 31, No. 7, pp. 346–350; July, 1925.

VEBLEN, O. Remarks on the foundations of geometry. Read Dec. 31, 1924. *This Bulletin*, vol. 31, Nos. 3–4, pp. 121–141; March–April, 1925.

VEBLEN, O., and THOMAS, J. M. Projective normal coordinates for the geometry of paths. Read May 2, 1925. *Proceedings of the National Academy of Sciences*, vol. 11, No. 4, pp. 204–207; April, 1925.

WAHLIN, G. E. On the solution of diophantine equations by means of ideals. Read Dec. 26, 1924. *This Bulletin*, vol. 31, No. 8, pp. 430–444; Oct., 1925.

WALSH, J. L. Some two-dimensional loci. Read Oct. 25, 1924. *Quarterly Journal of Pure and Applied Mathematics*, vol. 50, No. 2, pp. 154–165; Aug., 1924.

—— On Pellet's theorem concerning the roots of a polynomial. Read Dec. 27, 1923. *Annals of Mathematics*, (2), vol. 26, Nos. 1–2, pp. 59–64; Sept.–Dec., 1924.

WEDDERBURN, J. H. M. A theorem on simple algebras. Read May 3, 1924. *This Bulletin*, vol. 31, Nos. 1–2, pp. 11–13; Jan.–Feb., 1925.

—— The absolute value of the product of two matrices. Read May 2, 1925. *This Bulletin*, vol. 31, No. 7, pp. 304–308; July, 1925.

WEISNER, L. Groups in which the normaliser of every element except identity is abelian. Read Sept. 8, 1925. *This Bulletin*, vol. 31, No. 8, pp. 413–416; Oct., 1925.

—— On the number of elements in a group which have a power in a given conjugate set. Read Feb. 28, 1925. *This Bulletin*, vol. 31, Nos. 9–10, pp. 492–496; Nov.–Dec., 1925.

WIENER, N. The Dirichlet problem. Read May 3, 1924. *Journal of Mathematics and Physics of the Massachusetts Institute of Technology*, vol. 3, No. 3, pp. 127–147; April, 1924.

—— Note on a paper of O. Perron. Read Dec. 30, 1924. *Journal of Mathematics and Physics of the Massachusetts Institute of Technology*, vol. 4, No. 1, pp. 21–32; Jan., 1925.

—— A contribution to the theory of interpolation. Read Dec. 28, 1923. *Annals of Mathematics*, (2), vol. 26, No. 3, pp. 212–216; March, 1925.

—— The solution of a difference equation by trigonometrical integrals. Read Feb. 28, 1925. *Journal of Mathematics and Physics of the Massachusetts Institute of Technology*, vol. 4, No. 3, pp. 153–163; May, 1925.

—— Note on quasi-analytic functions. Read Dec. 30, 1924. *Journal of Mathematics and Physics of the Massachusetts Institute of Technology*, vol. 4, No. 4, pp. 193–199; July, 1925.

WILDER, R. L. A theorem on continua. Read May 3, 1924. *Fundamenta Mathematicae*, vol. 7, pp. 311–313; 1925.

—— Concerning continuous curves. Read April 15, 1922, Dec. 29, 1922, and Febr. 24, 1923. *Fundamenta Mathematicae*, vol. 7, pp. 340–377; 1925.

WILSON, W. A. On the Oscillation of a continuum at a point. Read Sept. 10, 1925. *Transactions of this Society*, vol. 27, No. 4, pp. 429–440; Oct., 1925.

WILSON, W. H. Two general functional equations. Read (Southwestern Section) Nov. 29, 1924. *This Bulletin*, vol. 31, No. 7, pp. 330–334; July, 1925.

WINGER, R. M. On the invariants of the ternary icosahedral group. Read (San Francisco Section) Dec. 22, 1923. *Mathematische Annalen*, vol. 93, Nos. 3–4, pp. 210–216; Jan., 1925.

ZELDIN, S. D. A classification of integral invariants in the calculus of variations. Read Dec. 27, 1923. *Tôhoku Mathematical Journal*, vol. 24, Nos. 1–2, pp. 121–124; Dec., 1924.

INDEX OF VOLUME XXXI

ALEXANDER, J. W. See REVIEWS, under Lefschetz.

BARNETT, I. A. Integro-Differential Equations, 263.

BATEMAN, H. See REVIEWS, under Jeans.

BELL, E. T. On the Number of Representations of an Integer as a Sum or Difference of Two Cubes, 309.

BENNETT, A. A. On Sets of Three Consecutive Integers which are Quadratic Residues of Primes, 411.

—— See REVIEWS, under Joly.

BERNSTEIN, B. A. Reports of Meetings of the San Francisco Section: October Meeting, 110; April Meeting, 385; June Meeting, 488.

BLISS, G. A. See TONELLI, L.

BROWN, B. H. See REVIEWS, under Fouët, Lemoyne.

CAMP, B. H. See REVIEWS, under Rietz.

CARMICHAEL, R. D. See REVIEWS, under Eddington, Halphen, Lamb, Lie, Nörlund.

CARSLAW, H. S. A Historical Note on Gibbs' Phenomenon in Fourier's Series and Integrals, 420.

COWLEY, E. B. See REVIEWS, under Loria.

CRATHORNE, A. R. See REVIEWS, under Borel, Fisher, Knopf, Urban.

CUMMINGS, L. D. A New Type of Double Sextette Closed under a Binary (3, 3) Correspondence, 266.

CURRIER, C. H. See REVIEWS, under Goodwin, Gribble.

DICKSON, L. E. Resolvent Sextics of Quintic Equations, 515.

—— See REVIEWS, under Fricke.

DODD, E. L. The Frequency Law of a Function of One Variable, 27.

—— See REVIEWS, under Forsyth.

DRESDEN, A. The International Congress at Toronto, 1.

—— Reports of Meetings of the American Mathematical Society: December Meeting in Chicago, 195; April Meeting in Chicago, 393; May Meeting in New York, 481.

—— See REVIEWS, under Bliss.

EMCH, A. See REVIEWS, under Bricard, Freyberger.

ETTLINGER, H. J. See REVIEWS, under Fubini, Kiepert, Knopp.

EVANS, G. C. Note on a Class of Harmonic Functions, 14.

—— See REVIEWS, under Enriques.

FIELD, P. See REVIEWS, under Montel.

FITE, W. B. Report of the February Meeting of the American Mathematical Society in New York, 291.

FORD, L. R. The Fundamental Region for a Fuchsian Group, 531.

FORSYTH, C. H. See REVIEWS, under Mills, Yule.

FRANKLIN, P. Functions with an Essential Singularity, 157.

FREEMAN, I. A General Form of the Suspension Bridge Catenary, 424.

GRAUSTEIN, W. C. See REVIEWS, under Coolidge.

GRIFFITHS, L. W. Contact Curves of the Rational Plane Cubic, 312.

HENDERSON, R. Life Insurance as a Social Service and as a Mathematical Problem, 227.

HILDEBRANDT, T. H. See REVIEWS, under Rothe.

HILLE, E. See REVIEWS, under Auerbach, Courant.

HOLLCROFT, T. R. Limits for Actual Double Points of Space Curves, 42.

HUNTINGTON, E. V. The Pan-American Scientific Congress, 290.

HURWITZ, W. A. Characteristic Parameter Values for an Integral Equation, 503.

JACKSON, D. The Geometry of Frequency Functions, 63.

KELLOGG, O. D. See REVIEWS, under Kneser.

KLINE, J. R. Concerning the Complementary Intervals of Countable Closed Sets, 409.

KURATOWSKI, C. On the Accessibility of an Arc from its Complement in Space of Three Dimensions, 32.

LEFSCHETZ, S. See REVIEWS, under de Kerékjártó.

LEHMER, D. N. A Correction, 226.

—— Note on the Construction of Tables of Linear Forms, 497.

—— See REVIEWS, under Cunningham, King, Mehmke, Whittaker.

LEVY, H. Normal Congruences of Curves in Riemann Space, 39.

—— Ricci's Coefficients of Rotation, 142.

LONGLEY, W. R. See REVIEWS, under Barton.

MCCLENON, R. B. See REVIEWS, under Caspar, Czwalina, Tropfke.

MACDUFFEE, C. C. On the Complete Independence of the Functional Equations of Involution, 21.

MACMILLAN, W. D. See REVIEWS, under Bieberbach.

MICHAL, A. D. Functional Invariants, with a Continuity of Order p, of One-Parameter Fredholm and Volterra Transformation Groups, 335.

MILLER, G. A. Simplifications relating to a Proof of Sylow's Theorem, 253.

MOORE, C. L. E. See REVIEWS, under Bertini, Schouten.

MOORE, C. N. Note on Gibbs' Phenomenon, 417.

—— See REVIEWS, under Hilb, Lichtenstein, Rosenthal.

MORGENSTERN, G. See REVIEWS, under Sakellarios.

MURNAGHAN, F. D. The Tensor Character of the Generalized Kronecker Symbol, 323.

—— See REVIEWS, under Sheppard.

OUSPENSKY, J. Sur les Valeurs Asymptotiques des Coefficients de Cotes, 145.

PERSONS, W. M. See REVIEWS, under Bowley.

PFEIFFER, G. A. See REVIEWS, under Fraenkel.

PORTER, M. B. See REVIEWS, under de la Vallée-Poussin.

REYNOLDS, C. N. See REVIEWS, under Henderson, Lecat, Lorentz.

RICHARDSON, R. G. D. Reports of Meetings of the American Mathematical Society: October Meeting in New York, 97; Thirty-First Annual Meeting, 203.

RICHARDSON, R. G. D. The Second Award of the Bôcher Memorial Prize, 193.

RITT, J. F. See REVIEWS, under Julia.

ROEVER, W. H. Some Phases of Descriptive Geometry, 540.

SHAW, J. B. See REVIEWS, under Hitchcock, Study.

SHOHAT, J. On the Polynomial of the Best Approximation to a Given Continuous Function, 509.

SHOOK, C. A. Improper Double Integrals, 524.

SNYDER, V. See REVIEWS, under Loria.

STONE, M. H. Three Theorems on Normal Orthogonal Sets, 17.

STOUFFER, E. B. Report of the Seventeenth Regular Meeting of the Southwestern Section, 116.

TAYLOR, J. H. Reduction of Euler's Equations to a Canonical Form, 257.

THOMAS, J. M. The Number of Even and Odd Absolute Permutations of n Letters, 303.

THOMAS, T. Y. Note on the Projective Geometry of Paths, 318.

TONELLI, L. The Calculus of Variations (translated by G. A. BLISS), 163.

UHLER, H. S. Criteria that any number of Real Points in n-Space shall lie in a $(n-k)$-Space, 499.

VANDIVER, H. S. On Sets of Three Consecutive Integers which are Quadratic or Cubic Residues of Primes, 33.

—— On the Distribution of Quadratic and Higher Residues, 346.

VEBLEN, O. Remarks on the Foundations of Geometry, 121.

WAHLIN, G. E. On the Solution of Diophantine Equations by Means of Ideals, 430.

—— See REVIEWS, under Bianchi, Cohen.

WEDDERBURN, J. H. M. A Theorem on Simple Algebras, 11.

—— The Absolute Value of the Product of Two Matrices, 304.

—— An Acknowledgement, 523.

—— See REVIEWS, under Lecat.

WEISNER, L. Groups in which the Normaliser of Every Element except Identity is Abelian, 413.

—— On the Number of Elements of a Group which have a Power in a Given Conjugate Set, 492.

WHITE, H. S. The Frank Nelson Cole Prize in Algebra, 289.

WIDDER, D. V. Some Mean-Value Theorems connected with Cotes's Method of Mechanical Quadrature, 56.

WILDER, C. E. See REVIEWS, under Galbrun.

WILSON, W. H. Two General Functional Equations, 330.

WINGER, R. M. Gauss and Non-Euclidean Geometry, 356.

WOODS, F. S. See REVIEWS, under Baker.

YOUNG, J. W. A. See REVIEWS, under Lietzmann.

Reviews

Archimedes. See Czwalina, A.

Auerbach, F. Die Methoden der theoretischen Physik, E. Hille, 555.

Baker, H. F. Principles of Geometry, volume II, F. S. Woods, 370.

Barton, E. H. Analytical Mechanics (second edition), W. R. Longley, 276.

Bertini, E. Einführung in die Projektive Geometrie Mehrdimensionaler Räume. Nach der zweiten italienischen Auflage deutsch herausgegeben von A. Duschek, C. L. E. Moore, 463.

Bianchi, L. Lezioni sulla Teoria dei Numeri Algebrici, G. E. Wahlin, 84.

Bieberbach, L. Theorie der Differentialgleichungen, W. D. MacMillan, 279.

Bliss, G. A. Calculus of Variations. Carus Mathematical Monographs, No. 1, A. Dresden, 551.

Borel, E. Eléments de la Théorie des Probabilités (troisième édition), A. R. Crathorne, 367.

—— Traité du Calcul des Probabilités et de ses Applications, tome III, fascicule 1: Assurances sur la Vie, par H. Galbrun, A. R. Crathorne, 466.

Borel, E., et Deltheil, R. Probabilités, Erreurs, A. R. Crathorne, 367.

Bowley, A. L. The Mathematical Groundwork of Economics, W. M. Persons, 469.

Bricard, R. Petit Traité de Perspective, A. Emch, 564.

Cahen, E. Théorie des Nombres. Tome second: Le Second Degré Binaire, G. E. Wahlin, 565.

Carus Mathematical Monographs. See Bliss, G. A.

Caspar, Max. Johannes Kepler. Mysterium Cosmographicum. Das Weltgeheimniss, R. B. McClenon, 560.

Coolidge, J. L. The Geometry of the Complex Domain, W. C. Graustein, 351.

Courant, R., und Hilbert, D. Methoden der Mathematischen Physik, erster Band, E. Hille, 456.

Cunningham, A. Binomial Factorizations, volumes I and IV, D. N. Lehmer, 180; volumes II, III, and VI, D. N. Lehmer, 462.

Czwalina, A. Ueber Paraboloide, Hyperboloide, und Ellipsoide von Archimedes, R. B. McClenon, 568.

Deltheil, R. See Borel, E.

Duschek, A. See Bertini, E.

Eddington, A. S. The Mathematical Theory of Relativity, (second edition), R. D. Carmichael, 563.

Encyklopädie der Mathematischen Wissenschaften. See Hilb, E., Lichtenstein, L., Rosenthal, A.

Engel, F. See Lie, S.

Enriques, F. Lezioni sulla Teoria Geometrica delle Equazioni e delle Funzioni Algebriche, volume I e volume II, G. C. Evans, 449.

Fisher, A. An Elementary Treatise on Frequency Curves and their Application in the Analysis of Death Curves and Life Tables. Translated from the Danish by E. A. Vigfusson, A. R. Crathorne, 275.

Flamant, P. See Julia, G.

Forsyth, C. H. Mathematical Theory of Life Insurance, E. L. Dodd, 371.
—— An Introduction to the Mathematical Analysis of Statistics, E. L. Dodd, 374.

Fouët, E. A. Leçons de Géométrie Elémentaire, B. H. Brown, 373.

Fraenkel, A. Einleitung in die Mengenlehre (zweite Auflage), G. A. Pfeiffer, 278.

Fréchet, M. See Rosenthal, A.

Freyberger, H. Zentral-Perspektive (zweite Auflage), herausgegeben von J. Vonderlinn, A. Emch, 88.

Fricke, R. Lehrbuch der Algebra, verfasst mit Benutzung von Heinrich Webers gleichnamigem Buche, erster Band, L. E. Dickson, 372.

Fubini, G. Lezioni di Analisi Matematica, H. J. Ettlinger, 74.

Galbrun, H. Introduction à la Théorie de la Relativité, Calcul Différentiel Absolu, et Géométrie, C. E. Wilder, 178.
—— See Borel, E.

Goodwin, H. B. The Alpha, Beta, Gamma Navigation Tables, C. H. Currier, 88.

Gribble, T. G. Handy Methods of Geodetic Astronomy for Land Surveyors, C. H. Currier, 87.

Halphen, G. H. Oeuvres, tome IV, R. D. Carmichael, 562.

Heegaard, P. See Lie, S.

Henderson, A., Hobbs, A. W., and Lasley, J. W. The Theory of Relativity, Studies and Contributions, C. N. Reynolds, 277.

Hilb, E., und Riesz, M. Neuere Untersuchungen über Trigonometrische Reihen. Sonderabdruck aus der Encyklopädie der Mathematischen Wissenschaften, C. N. Moore, 453.

Hilbert, D. See Courant, R.

Hitchcock, F. L., and Robinson, C. S. Differential Equations in Applied Chemistry, J. B. Shaw, 87.

Hobbs, A. W. See Henderson, A.

Jeans, J. H. Report on Radiation and the Quantum-Theory, H. Bateman, 85.

Joly, G. Les Erreurs Philosophiques de M. Einstein. Etude Directe de la Relativité, A. A. Bennett, 567.

Julia, G. Leçons sur les Fonctions Uniformes à Point Singulier Essentiel Isolé, rédigées par P. Flamant, J. F. Ritt, 359.

Kepler, J. See Caspar, M.

de Kerékjártó, B. Vorlesungen über Topologie, erster Band, S. Lefschetz, 176.

Kiepert, L. Grundriss der Differential-Rechnung, zweiter Band, H. J. Ettlinger, 74.

King, L. V. On the Direct Numerical Calculation of Elliptic Functions and Integrals, D. N. Lehmer, 568.

Kneser, A. Die Integralgleichungen und ihre Anwendungen in der Mathematischen Physik (zweite Auflage), O. D. Kellogg, 177.

Knopf, O. Wahrscheinlichkeitsrechnung, A. R. Crathorne, 367.

Knopp, K. Aufgabensammlung zur Funktionentheorie, erster Teil, H. J. Ettlinger, 183.

Lamb, H. Hydrodynamics (fifth edition), R. D. Carmichael, 460.

Lasley, J. W. See Henderson, A.

Lecat, M. Bibliographie des Déterminants à Plus de Deux Dimensions, J. H. M. Wedderburn, 86.

—— Bibliographie de la Relativité, C. N. Reynolds, 566.

Lefschetz, S. L'Analysis Situs et la Géométrie Algébrique, J. W. Alexander, 558.

Lemoyne, T. Les Lieux Géométriques en Mathématiques Spéciales, B. H. Brown, 182.

Lichtenstein, L. Neuere Entwicklung der Theorie Partieller Differentialgleichungen zweiter Ordnung vom Elliptischen Typus. Sonderabdruck aus der Encyklopädie der Mathematischen Wissenschaften, C. N. Moore, 453.

Lie, S. Gesammelte Abhandlungen, herausgegeben von F. Engel und P. Heegaard, fünfter Band, erste Abteilung, R. D. Carmichael, 559.

Lietzmann, W. Methodik des Mathematischen Unterrichts, dritter Teil, J. W. A. Young, 374.

Lorentz, H. A. Clerk Maxwell's Electromagnetic Theory, C. N. Reynolds, 277.

Loria, G. Complementi di Geometria Descrittiva, E. B. Cowley, 466.

—— Curve Sgembe Speciali Algebriche e Transcendenti. Volume primo: Curve Algebriche, V. Snyder, 557.

Mehmke, R. Leitfaden zum Graphischen Rechnen (zweite Auflage), D. N. Lehmer, 467.

Montel, P. Statique et Résistance des Matériaux, P. Field, 180.

—— See Rosenthal, A.

Mills, F. C. Statistical Methods, C. H. Forsyth, 563.

National Research Council. See Rietz, H. L.

Nörlund, N. E. Vorlesungen über Differenzenrechnung, R. D. Carmichael, 445.

Riesz, M. See Hilb, M.

Rietz, H. L. (editor). Handbook of Mathematical Statistics, prepared by a committee of the National Research Council, B. H. Camp, 361.

Robinson, C. S. See Hitchcock, F. L.

Robinson, G. See Whittaker, E. T.

Rosenthal, A. Neuere Untersuchungen über Funktionen Reeller Veränderlichen, nach den Referaten von L. Zoretti, P. Montel, und M. Fréchet. Sonderabdruck aus der Encyklopädie der Mathematischen Wissenschaften, C. N. Moore, 453.

Rothe, R. Höhere Mathematik. Teil I: Differentialrechnung und Grundformeln der Integralrechnung, nebst Anwendungen, T. H. Hildebrandt, 566.

Sakellarios, N. Elements of Analytic Geometry, volume I (in Greek), G. Morgenstern, 465.

Schouten, J. A. Der Ricci-Kalkül, C. L. E. Moore, 173.

Sheppard, W. F. From Determinant to Tensor, F. D. Murnaghan, 468.

Study, E. Einleitung in die Theorie der Invarianten Linearer Transformationen auf Grund der Vektorenrechnung, erster Teil, J. B. SHAW, 77.

Tropfke, J. Geschichte der Elementarmathematik, fünfter und sechster Band, R. B. MCCLENON, 369; siebenter Band, R. B. MCCLENON, 461.

Urban, F. M. Grundlagen der Wahrscheinlichkeitsrechnung und der Theorie der Beobachtungsfehler, A. R. CRATHORNE, 367.

Vigfusson, E. A. See Fisher, A.

de la Vallée Poussin, C. J. Cours d'Analyse Infinitésimale (cinquième édition), volume I, M. B. PORTER, 83.

Vonderlinn, J. See Freyberger, H.

Weber, H. See Fricke, R.

Whittaker, E. T., and Robinson, G. The Calculus of Observations, D. N. LEHMER, 181.

Yule, G. U. An Introduction to the Theory of Statistics, (seventh edition), C. H. FORSYTH, 568.

Zoretti, L. See Rosenthal, A.

Errata, 226.

Index of Volume XXXI, 589.

New Publications, 93, 189, 285, 379, 476, 575.

Notes, 89, 184, 280, 375, 470, 569.

Papers Read Before the Society and Subsequently Published, Thirty-Fourth Annual List of, 579.

NOTES AND OTHER ITEMS

Academies, Associations, Congresses, and Societies:

American Mathematical Society: Annual Meeting, 206; Auditors' Report, 205; Bôcher Prize, 193, 207; Budget, 205; Bulletin, 569; Cole Prize, 102, 289; Colloquium, 375, 396; Committee on Gibbs Lecture, 395; on Membership, 207; on Moore Fund, 101; on Printing, 396; Election of Officers, 206; Endowment Fund, 101, 205, 395; Gibbs Lecture, 101, 207, 375, 395; Librarian's Report, 206; New Members Admitted, 97, 98, 204, 393; Representatives on Council of American Association for the Advancement of Science, 206, 280; at International Mathematical Congress, 101; on National Research Council, 206, 375; at Pan-American Scientific Congress, 101; San Francisco Section, 111, 385, 488; Southwestern Section, 116, 375; Statistics, 205; Summer Meeting, 375, 396; Sustaining Memberships, 97, 101, 204, 393; Transactions, 89, 184, 375, 470, 569; Treasurer's Report, 205; Western Christmas Meeting, 375, 396.

Associations for the Advancement of Science; American, 206, 280; 375, 376; British, 89, 185, 376.

Académie des Sciences de l'Oukraine, 281; American Astronomical Society, 375; American Physical Society, 375; Belgian Academy, 376, 471; Bologna Academy, 282; Boston Society of Civil Engineers, 571; Carnegie Institution, 376; Charkow Mathematical Society, 89; Copenhagen Society for the Spread of Natural Philosophy, 376;

Engineering Foundation, Carnot Celebration, 185; Franklin Institute, 185, 571; General Electric Company, 185; Institut de France, 282; International Astronomical Union, 281; International Mathematical Congress, 1, 101, 185; International Mathematical Union, 2, 3; International Physics Union, 281; International Research Council, 281; London Institution of Electrical Engineers, 376; London Mathematical Society, 280; Mathematical Association of America: Carus Monographs, 184; Meeting, 375; Officers, 280; Naples Academy, 570; National Research Council, 90, 206, 375; National Research Council of Japan, 570; Pan-American Scientific Congress, 101, 186, 290, 377; Paris Academy of Sciences, 282, 471; Royal Astronomical Society, 376; Royal Society of Edinburgh, 570; Russian Information Bureau, 281; Sociedad Matemática Argentina, 570; Società Universitaria di Scienze Matematiche, 89; Société Mathématique de France, 281; Unione Matematica Italiana, 282; Westphalian Mathematical Society, Weierstrass Week, 376.

Books, Announcement of New, 184, 281, 376.

Doctorates in Mathematics, American, 472.

Fellowships: American Field Service, 471; Guggenheim, 471,572; National Research, 572.

Journals:

Acta Literarum ac Scientiarum, Szeged, 281, 291; American Journal of Mathematics, 89, 184, 280, 570; Annales Scientifiques des Institutions Savantes de l'Oukraine, Sect. Math., 89; Annals of Mathematics, 89, 280, 569; Archimede, 89; Astrophysical Journal, 376; Bulletin de l'Académie des Sciences de l'Oukraine, 281; This Bulletin, 569; Communications of the Mathematical Society of Charkow, 89; Japanese Journal of Mathematics, 570; Mémorial des Sciences Mathématiques, 281; Revista Matemática, 570; Transactions of the American Mathematical Society, 89, 184, 375, 470, 569.

Papers and Communications Presented to the Society, Authors:

Adams, C. R., 216, 217; Alexander, J. W., 106, 299, 299; Allen, E. S., 119.

Babcock, R. W., 404; Ballantine, J. P., 487; Barnett, I. A., 201; Barter, J., 385; Bateman, H., 111, 386; Bell, E. T., 111, 111, 112, 112, 113, 113, 386, 488, 488; Bender, H. A., 400, 400; Bennett, A. A., 223, 223, 224, 224, 296, 296, 399; Bernstein, B. A., 113, 386, 488; Betz, H., 200; Birkhoff, G. D., 116; Blichfeldt, H. F., 386; Bliss, G. A., 215.

Cajori, F., 113, 387, 387, 489; Camp, B. H., 102; Camp, C. C., 402; Caris, P. A., 399; Carpenter, A. F., 387, 489; Carson, J. R., 481; Castellani, M., 483; Chittenden, E. W., 117, 223; Church, A., 226, 485; Cole, F. N., 103; Colpitts, J. T., 118; Cramlet, C. M., 489; Crane, R., 397; Cummings, L. D., 102.

Daus, P. H., 387; Davis, H. T., 200; Dickson, L. E., 198, 199; Dodd, E. L., 200, 225, 296.

Edington, W. E., 200; Eisenhart, L. P., 292; Emch, A., 398; Ettlinger, H. J., 201.

Foster, R. M., 222; Franklin, P., 103, 103, 225, 301, 482, 482; Frink, O., 485.

Garabedian, C. A., 104, 298; Gehman, H. M., 103, 220, 301;

Glenn, O. E., 213, 483; Graustein, W. C., 212, 213; Graves, L. M., 211, 211; Green, C. F., 119; Gronwall, T. H., 109, 109, 302, 481, 484, 484; Grove, V. G., 196, 397.

Hancock, H., 207; Haskell, M. W., 114, 388; Hazlett, O. C., 219, 482; Hedrick, E. R., 114, 218, 388; Henderson, A., 207; Henderson, R., 207; Hille, C. E., 104, 107, 302; Hollcroft, T. R., 482; Hotelling, H., 114, 202, 389, 389; Huber, C. M., 105; Huntington, E. V., 405; Hurwitz, W. A., 215, 215, 216; Hutchinson, J. I., 104.

Ingold, L., 114, 119, 201, 294, 407, 407; Ingraham, M. H., 405, 408.

Jackson, D., 107, 109, 403, 404; James, G., 298, 389; Jeffery, R. L., 299; Jenkins, W. A., 407.

Kasner, E., 214, 226, 293; Kellogg, O. D., 300; Kempner, A. J., 201; de Kerékjártó, B., 302; Kline, J. R., 110, 300.

Lane, E. P., 196; Langer, R. E., 301; Latimer, C. G., 213; Lefschetz, S., 301; Lehmer, D. N., 389, 489; Lehr, A. M. M., 209; Levy, H., 294, 294, 485; Libman, E. E., 297, 398; Lubben, R. G., 225.

McDonald, J. H., 390, 390, 391, 391, 391; MacDuffee, C. C., 400; MacMillan, W. D., 396; MacNeish, H. F., 198, 212; March, H. W., 405; Mathews, R. M., 397; Merriman, G. M., 210; Michal, A. D., 106; Miller, G. A., 199, 401; Moore, C. L. E., 292; Moore, C. N., 210; Moore, R. L., 120, 120, 219, 220, 220; Moritz, R. E., 391, 490, 490; Morley, F., 293; Morris, F. R., 115; Morse, H. M., 216; Mullemeister, H., 392; Murray, F. H., 221; Musselman, J. R., 212;

Olson, H. L., 295; Osgood, W. F., 295.

Pell, A. J., 210; Pennell, W. O., 106; Perkins, F. W., 221; Pierce, T. A., 119.

Rainich, G. Y., 108, 109, 209, 209, 292, 293; Reilly, J. F., 120, 399; Reimer, E. H., 115; Reynolds, C. N., 297; Rider, P. R., 197; Ritt, J. F., 107, 108, 218, 300; Robertson, H. P., 115, 491; Robinson, G. P., 408; Roever, W. H., 116; Russell, H. N., 207.

Sharpe, F. R., 214; Shaw, J. B., 202, 406; Shohat, J. A., 197, 402; Smith, H. L., 403; Stager, H. W., 491; Stone, M. H., 225, 225; Synge, J. L., 214.

Taylor, J. H., 102, 487; Taylor, J. S., 222; Thomas, J. M., 299, 299, 486, 486, 486; Turner, J. S., 118, 118, 408.

de la Vallée Poussin, C. J., 219; Vandiver, H. S., 208, 208, 208; Veblen, O., 207, 486.

Wahlin, G. E., 199, 401; Walsh, J. L., 105; Wedderburn, J. H. M., 484, 484; Weisner, L., 299; Wiener, N., 106, 221, 221, 221, 302, 481, 482, 484; Wilder, R. L., 224; Williams, K. P., 105, 402; Wilson, W. H., 117, 117, 404; Winger, R. M., 491; Wood, F. E., 195.

Zeldin, S. D., 212.

Personal Notes:

Abrich, G. F., 186; Adams, C. R., 474; Agnew, R. P., 475; Albanese, G., 377; Allen, H. S., 570; Amaldi, M., 90; Ames, J. S., 90, 375, 375; Ames, L. D., 574; Anderson, A. E., 92; Andoyer, H., 185; Andreoli, G., 377; Appell, P., 471, 471, 471; d'Arcais, F., 283; Archibald, R. C., 280, 280; Armstrong, C. N., 572; Arnold, K. S., 91; Arnold, S. T., 378; Austin, C. B., 188, 572; Ayres, F., 284.

Babcock, R. W., 472; Bailey, E. A., 474; Baker, B. B., 90; Ball, W. W. R., 475; Balof, C. A., 92; Bateman, H., 282; Bauer, G. N., 186; Baxter, E. F., 91, 572; Beckwith, E. R., 91; Beghin, H., 283; Behnke, H., 378; Belardinelli, G., 283; Bell, E. T., 193, 474; Beloch, M. P., 378; Bennett, A. A., 280, 378; Bennett, J. C., 92; Benson, C. H., 378; Bernstein, S., 89; Berryman, C. H., 188; Berzolari, L., 283; Betz, H., 472; Bianchi, L., 282; Bickerton, A. W., 282; Bieberbach, L., 376; Bill, E. G., 186; Birkhoff, G. D., 375, 375, 474; Bloch, P., 282; Blumberg, A. A., 284; Blumberg, H., 474; Blumenthal, L. M., 378; Bohr, N., 282, 376, 471, 571, 571; Bowie, W., 375; Boyd, E. E., 188; Brackett, F. P., 186; Bradley, F. H., 188; Bragg, W. H., 89, 185; Bray, H. E., 91; Brinkmann, H. W., 475; Brodetsky, S., 90: Bruce, D., 185; Bruton, G. S., 188; Bubb, F. W., 186; Buchanan, M., 572; Burgess, R. W., 474; Byrne, W. E., 471.

Caldonazzo, B., 377; Campbell, A. D., 472; Campbell, G. A., 185; Campbell, J. E., 188; Campbell, M., 186; Carlen, M., 574; Carlson, E., 473; Carnot, S., 185; Carroll, E. T., 91; Cartan, E., 283; Carter, B. E., 284; Carver, W. B., 474; Cassina, U., 283; Chambers, G. G., 571; Chaney, J. C., 188; Chapman, S., 91; Chazy, J., 472; Cherry, T. M., 91; Ciani, E., 283; Coker, E. C., 91; Collignon, G. H., 473; Colpitts, J. T., 473; Comrie, L. J., 187; Conrad, W. A., 284; Coolidge, J. L., 280; Crathorne, A. R., 91; Crozet-Fourneyron, M., 282; Crudeli U., 377.

Dale, J., 473, 572; Daniele, E., 283; Darkow, M. D., 473; Davis, H. A., 574; Davis, J. E., 573; De Long, I. M., 91; Dempster, A. J., 375; Dennett, W. S., 574; De Ronde, L. A., 186; Dixon, A. L., 280; Doetsch, G., 186; de Donder, T., 376; Doshkees, V., 187; Dresden, A., 206, 280; Dyson, F., 376.

Eddington, A. S., 90, 472, 571, 571; Eglin, W. C. L., 185; Einstein, A., 186, 471; Eisenhart, L. P., 280, 283, 375, 375; Elliott, W. W., 473; Eversull, B. M., 473.

Fabry, C., 185, 282; Farnum, F., 475; Fellar, W. L., 472; Fermi, E., 378; Ferrar, W. L., 572; Fichot, E., 472; Filon, L. N. G., 280; Finzi, A., 283; Flammarion, C., 574; Föppl, A., 188; Foley, A. L., 375; Foote, P. D., 375; Fourneyron, B., 282; Fowler, R. H., 471, 472; Franklin, P., 573; Freeman, J. R., 571; Fubini, G., 282; Fulmer, H. K., 187.

Gans, R., 472; Garabedian, C. A., 474; Garabedian, H. L., 475; Garnier, R., 282; Gaylord, L., 91; Gerst, F. J., 473; Gibbens, G., 573; Giddings, H. A., 284; Gigli, D., 90; Gilchrist, L., 91; Gillespie, D. C., 474; Giraud, G., 282; Gleason, R. E., 283; Gossard, H. C., 573; Goursat, E., 282; Gouwens, C., 473; Graves, L. M., 473, 475; Green, W. C., 188; Grove, C. C., 573; Grumann, H. R., 188; Guichard, C., 282, 475; Gummer, C. F., 473; Gutzmer, A., 92.

Hadamard, J., 377; Hadlock, E. H., 188; Hale, G. E., 90; Hall, A. G., 284; Hamilton, W. A., 574: Hammatt, H. B., 475; Hammerstein, A., 90; Hardy, G. H., 280; Harkins, C., 574; Hart, W. L., 573; Haudié, E., 282; Hazlett, O. C., 573; Heaviside, O., 475; Hedley, J. W., 473; Heinke, P. E., 284; Hemke, P. E., 473; Herbert, W. C., 188; Herglotz, G., 377; Hermite, C., 185; Hess, G. W., 187; Highet, H. D., 188; Hilbert, D., 376; Hill, M. J. M., 283; Hill, W. H., 91; Hilton, H., 280; Hinsch, V. B., 378; Hobbs, A. W., 378; Holley, J. L., 473; Hoover, H. A., 92; Hotelling, H., 473; Householder,

F. W., 284; Howie, J. N., 92; Hughes, J. C., 473; Hunt, M., 473; Huntington, E. V., 186, 280, 377, 572; Hussey, L. W., 188.

Ingraham, M. H., 473.

Jackson, D., 280, 375; Janet, B., 377; Jeans, J. H., 92, 472; Jeffreys, H., 472; Jenkins, W. A., 475; Jensen, C. M., 473; Johnson, W. A., 281; Jonah, F. C., 188; Jones, E. M., 185; Jones, J. E., 472; Jones, R. L., 188; Jude, R. H., 92; Julia, G., 472.

von Kármán, T., 572; Kells, L. M., 284; Kempner, A. J., 573; de Kerékjártó, B., 291; Kiernan, A., 284; Klein, F., 90, 376, 574; Kline, J. R., 572; Kneser, A., 571; Knopp, K., 376; von Koch, H., 188; Koebe, P., 376, 571; Koenigs, G., 185; Kohlschütter, A., 376; Koopman, B. O., 475; Kotof, T., 92; Krauss, F., 378; Kryloff, N. M., 281.

Lamb, H., 89; Lambert, P. A., 187, 475; Lane, H. L., 475; Langevin, P., 282; Langmuir, I., 572; Larkin, E. L., 188; Lasley, J. W., 378; Latimer, C. G., 473; von Laue, M., 282; Lazzarino, O., 283; Lecornu, L., 472; Lefschetz, S., 193, 284, 571; Lehmer, D. N., 186, 376; Lehr, A. M. M., 92; Leudesdorf, C., 92; Leuschner, A. O., 377; Levy, H., 473; Linker, J. B., 378, 474; Lockner, S. J., 574; Longbottom, J. G., 92; Loring, W. E., 284; Lund, E. H., 188; Lyon, W. H., 187.

McAdam, D. J., 475; McClintick, W. S., 378; McFarlan, L. H., 474; McLaughlin, D. B., 188; McLennan, J. C., 90; MacClaren, M., 475; MacPherson, D. H., 92, 284; Maggi, G., 283; Malo, E., 282; Marden, M., 475; Marquis, R. H., 188; Martinson, E. P., 92; Maslow, A. P., 188; Mason, M., 573; Menzel, D. H., 92; Mercer, E., 188; Merrill, A. S., 92; Merrill, J. E., 188; Merriman, M., 574; Meyer, F., 378; Michal, A. D., 474; Michelson, A. A., 90, 186, 186, 573; Michie, J. N., 475; Miles, E. J., 573; Miller, D. C., 375; Millikan, R. A., 90, 90, 572; Milne, E. A., 91; Miser, W. L., 475; Mitchell, H. B., 187; Mittag-Leffler, G., 376; Montel, P., 282, 472; Montgomery, R. B., 187; Moorefield, N. L., 188; Morris, I., 187; Morris, M., 573; Morse, H. M., 378; Müller-Breslau, H., 574.

Neder, L., 377; Nekrassoff, P. A., 475; Neubauer, G., 574; Neumann, C., 574; Nicholson, G. W., 188; Nordgaard, M., 186; Nyswander, J. A., 474.

Olson, H. L., 378; Osborn, J. O., 187; Ott, W. P., 187.

Page, L., 375; Painlevé, P., 571; Palatini, A., 283; Palmié, A. H., 91; Parsons, C., 185, 185; Patton, L. K., 188; Pell, A. J., 91, 573; Perkins, F. W., 475; Perron, C., 376; Picard, E., 185, 282, 471; Picone, M., 283; Pierpont, J., 375; Pixsley, H. H., 188; Plapp, E. M., 574; Poincaré, H., 471; Pollard, H. S., 92; Pólya, G., 377; Porter, C. S., 92; Priulla, G., 89; Proudman, J., 472; Pugh, G. T., 187; Pupin, M. I., 90, 280, 572; Putnam, R. G., 474; Putnam, W. L., 284.

Rademacher, H., 472; Rainich, G. Y., 470; Ramler, O. J., 187; Rea, P. L., 187; Rechard, O. H., 573; Reeve, W. D., 475; Reinsch, B. P., 474; Reynolds, C. N., 572; Rice, E. W., 185; Richardson, D. P., 188; Richardson, R. G. D., 284; Richtmyer, F. K., 375, 375; Riesz, F., 281; Rietz, H. L., 280; Riquier, C., 283; Ritt, J. F., 470; de la Rive, L., 92; Roberts, S. E., 475; Robinson, G. E., 92; Roever, W. H., 280; Romboli, G., 574; Room, T. G., 471; Rowe, J. E., 187; Rowland, S. A., 186; Runge, C., 378; Rupp, C. A., 187; Russell, B., 572; Russell, J., 185; Rutherford, E., 90, 185.

Sannia, G., 283; Sansone, G., 283; Saund, D. S., 474; Schafheitlin, P., 378; Schrutka, L., 472; Scott, C. A., 91; von Seeliger, H., 284; Severi, F., 185; Shaw, E. A., 284; Sherwood, G. E. F., 378; Shiefer, H. F., 378; Shohat, J. A., 188, 573; Silla, L., 90; Simpson, C. G., 376; Smail, L. L., 573; Smith, D. E., 475; Smith, P. K., 284; Solum, A., 186; Sparks, F. W., 284; Sperry, C. S., 475; Stabler, H. P., 475; Stafford, E. T., 188, 574; Stager, H. W., 377; Stanton, E. W., 284; Starke, E. P., 574; Stebbins, J., 375; Steinert, E. E., 188; Steinmetz, C. P., 185; Stekloff, V., 185; Stevens, W. P., 284; Stokes, N. C., 92; Stone, M. H., 378; Stouffer, E. B., 206, 280; Straneo, P., 377; Stromquist, C. E. 574; Symonds, B., 284; Synge, J. L., 572; Synnerdahl, J. H., 187.

Talley, J. L., 188; Taylor, J. H., 474, 574; Taylor, J. S., 187; Terracini, A., 377; Thomson, E., 571; Thomson, J. J., 376, 570; Tienzo, T., 378; Tietze, H., 378; Tinner, J. C., 92; Tisdale, V., 187; Tisdale, W. E., 375; Togliatti, C., 472; Torelli, G., 283; Torrey, M. M., 474; Tricomi, F., 377; Turner, B. M., 572; Tyndall, A. M., 376.

Udinski, W. P., 284; Uhler, H. S., 574; Urysohn, P., 378.

de la Vallée Poussin, C. J., 185; Van Fleet, G. S., 188; Vassilief, A. V., 185, 571; Veronese, G., 571; Vessiot, E., 282; Villat, H., 281; Vitali, G., 283; Vivanti, G., 283; Voss, A., 571.

Wagner, E. H., 378; Wallenberg, G. J., 378; Walsh, J. L., 474, 572; Walther, A., 378; Watson, G. N., 280; Wavre, R., 90; Webster, A. G., 90; Webster, D. L., 375; Weitzenböck, R., 185; Wells, E. D., 188; Wellstein, J., 378; Western, A. E., 280; Wheeler, A. L., 573; Whitehead, A. N., 91, 186; Whitford, D. E., 475; Whittaker, E. T., 283; Whyburn, W. M., 92; Widder, D. V., 91, 474; Wieghardt, K., 188; Wien, W., 471; Wiggin, E. P., 92; Wilder, R. L., 188; Wiley, F. B., 475; Williams, W. L., 188; Wilson, E. B., 280; Wilson, H. A., 186, 574; Wilson, L. T., 475; Winsor, A. S., 378; Wirtinger, W., 571; Wolfender, J., 475; Wood, F., 474, 574.

Young, W. H., 280.

Zeeman, P., 185, 571; Zeiss, C., 376; Ziwet, A., 475.

Prizes and Medals:

Abbe, 376; Ackermann-Teubner, 376; Arzelà, 282; Barnard, 471; Belgian Academy, 376, 471; Bôcher, 193, 207; Cambridge University: Seely, 281, Adams, 471, Smith's, 471; Cole, 102, 289; Faraday, 376; Franklin Institute, 571; Freeman, 571; Fubini, 282; Lubbock, 185; Oersted, 376; Osiris, 282; Paris Academy, 282; Poincaré, 471; Royal Astronomical Society, 376; Royal Society of Edinburgh, 570; Torelli, 570.

Universities and Technical Schools:

Cambridge, 281, 471;
Columbia, 470, 471;
Harvard, 90, 470;
London, 185;
Oxford, 571;
Rensselaer Polytech Institute, 185;
Szeged, 281;
Union, 185.